Arithmetic Groups, lecture 1 10.4.2018

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D. Witte-Morris: Introduction to Arithmetic Groups

oral exam

some exercises will be presented to us, highly recommended to try to solve them

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Idea: groups obtained as integer points of algebraic groups

Ex. $SL_n(\mathbb{Z}) = \{ A \in Mat_{n\times n}(\mathbb{Z}) \mid \det(A) = 1 \}$

Def. $\mathbb{H}^2 := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}^2$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$ easily seen to be well-defined

Need to check: this action preserves $\mathbb{H}^2$:

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)$$

$$= \frac{1}{2i} \cdot \frac{(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)}{|cz+d|^2}$$

$$= \frac{1}{2i} \cdot \frac{\overbrace{(ad-bc)}^{\wedge} z - \overbrace{(ad-bc)}^{\wedge} \bar{z}}{|cz+d|^2} = \frac{\mathrm{Im}(z)}{|cz+d|^2} > 0$$

The interesting thing is that this action preserves the metric as well.

We can define a metric on $\mathbb{H}^2$ (and obtain the hyperbolic geometry) by:

$$g_{x+iy}: T_{x+iy}\mathbb{H}^2 \times T_{x+iy}\mathbb{H}^2 \longrightarrow \mathbb{R}$$

$$g_{x+iy}(u,v) = \frac{1}{y^2}\langle u, v \rangle$$

One notes that while distances differ from the Euclidean ones, angles do not.

Claim. $SL_2(\mathbb{Z})$ preserves g.

Pf: EXERCISE. (Riemannian geometry)

Def. $(\mathbb{H}^2, g)$ hyperbolic plane.

Question. What kind of space is $\mathbb{H}^2/SL_2(\mathbb{Z})$?

Def. Fundamental domain: if X is a top. space, G a group, $G \curvearrowright X$ by homeomorphisms, then a fund. domain for this action is a closed set

$F \subseteq X$ s.t. (1) $\bigcup_{g \in G} gF = X$

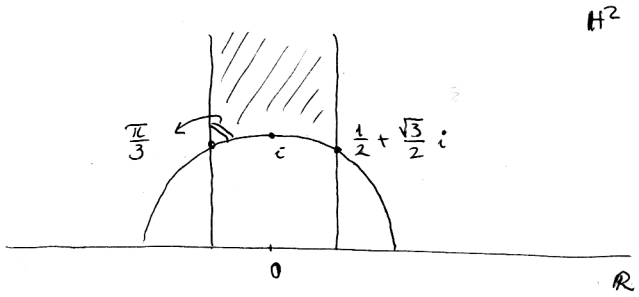
(2) $g\mathring{F} \cap \mathring{F} = \emptyset \quad \forall g \neq e \in G$

Ex. (0) $\mathbb{Z} \curvearrowright \mathbb{R}$ by $m \cdot x = x + m$ $\longrightarrow$ fund. domain: $[0,1]$

$$\mathbb{R}/\mathbb{Z} = S^1 = [0,1]/_{0 \sim 1}$$

Claim. $\mathcal{F} = \left\{ z \in \mathbb{H}^2 \;\middle|\; |z| \geq 1,\ -\frac{1}{2} \leq \mathrm{Re}\, z \leq +\frac{1}{2} \right\}$ is a fund. domain for $SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}^2$.

Pf: EXERCISE 1.3.7. & 1.3.8. in [WH].



$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{elts of } SL_2(\mathbb{Z})$$

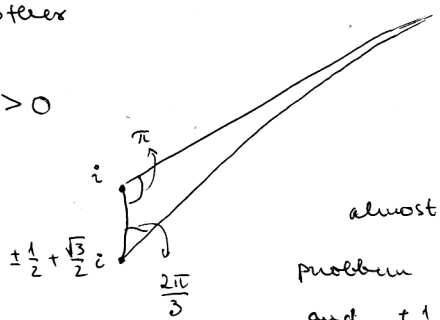
$$Sz = \frac{-1}{z} \qquad Tz = z + 1$$

The action of S on the circle:

$S(e^{i\vartheta}) = -e^{-i\vartheta} = e^{i(\pi - \vartheta)} \Rightarrow S(i) = i$ and the arcs ⌒ and ⌒ get mapped to each other

$$T\left(-\tfrac{1}{2} + ai\right) = \tfrac{1}{2} + ai \quad \forall a > 0$$

Claim. The quotient is



→ "orbifold"

almost a manifold,

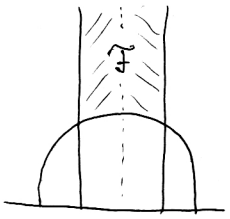
problem with the tg. spaces at i and $\pm\frac{1}{2} + \frac{\sqrt{3}}{2} i$.

(These come from the facts $S^2 = \mathrm{Id}$ and $(ST)^3 = \mathrm{Id}$.)

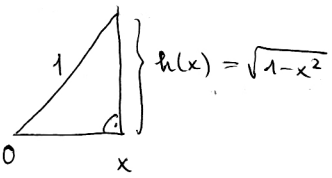
Fact. $\exists$ finite cover of $\left(\mathbb{H}^2/SL_2(\mathbb{Z})\right)\setminus$ problematic points (that is, a branched cover of $\mathbb{H}^2/SL_2(\mathbb{Z})$) that is a manifold.

The quotient has finite area:

$$\text{area}\left(\mathbb{H}^2/SL_2(\mathbb{Z})\right) = \int_{\mathcal{F}} \frac{dx\,dy}{y^2} = 2\int_0^{1/2}\int_{h(x)}^{\infty} \frac{dx\,dy}{y^2}$$



$$= 2\int_0^{1/2} \left[-\frac{1}{y}\right]_{\sqrt{1-x^2}}^{\infty} dx$$



$$= 2\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = 2\left[\arcsin x\right]_0^{1/2}$$

$$= 2\cdot\frac{\pi}{6} = \frac{\pi}{3} < \infty$$

Summary. Input: arithmetic subgroup of a Lie group (ie. $SL_2(\mathbb{R})$ in our case)

Output: finite volume "nice space" with a Riemannian metric coming from a symmetric space the group acts on.

(↑ this means sth. nice)

locally symmetric

This course is about properly understanding and generalizing the above.

- Locally symmetric spaces
- Lattices

} "vocabulary" for the topics that follow

- Arithmetic groups: def., examples, basic properties
- Advanced topics:
  - Margulis' super rigidity
  - Non-arithmetic lattices in $SO(n,1)$
  - Arithmetic groups coming from hypergeometric functions (i.e. solutions to the hypergeometric DE)

1. Locally symmetric spaces

§ 1. Symmetric spaces

Def. A Riemannian mf. (M, g) is **homogeneous** if the gp of isometries

$$\mathrm{Isom}(M) = \{\varphi : M \to M \text{ diffeo.}, \varphi^* g = g\}$$

acts transitively on M, i.e. $\forall x, y \in M \ \exists \varphi \in \mathrm{Isom}(M) : \varphi(x) = y$.

Ex. $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ w/ metric from $\mathbb{R}^{n+1}$

$$\mathrm{Isom}(S^n)^+ = \{\varphi \in \mathrm{Isom}(S^n) \mid \varphi \text{ preserves orientation}\}$$

$$= SO(n+1) = \{A \in \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{R}), \langle Av, Aw\rangle = \langle v, w\rangle, \det A = 1\}$$

($\langle\cdot,\cdot\rangle$: standard inner product on $\mathbb{R}^{n+1}$)

We can rotate any point of S^n to any other point, rotations are in $SO \Rightarrow S^n$ is homogeneous.

Ex. $\mathbb{R}^n$ is homogeneous with the std. Eu. metric

$$\mathrm{Isom}^+(\mathbb{R}^n) = \underbrace{SO(n)}_{\text{rotations}} \ltimes \underbrace{\mathbb{R}^n}_{\text{translations}}$$

Recall the notion of the semidirect product: N, H groups, $\varphi : H \to \mathrm{Aut}(N)$ a homomorphism. Then one defines $N \underset{\varphi}{\rtimes} H$ as follows:

the underlying set is the Cartesian product $N \times H$;

the multiplication is given by:

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2).$$

In the above case: $A_1, A_2 \in SO(n)$, $v_1, v_2 \in \mathbb{R}^n$

$$(A_1, v_1) \cdot (A_2, v_2)\, w = (A_1, v_1)(A_2 w + v_2) = A_1 A_2 w + A_1 v_2 + v_1,$$

i.e. we have $\varphi : SO(n) \to \mathrm{Aut}(\mathbb{R}^n)$ just the action of $SO(n)$ on $\mathbb{R}^n$.

Nb. translations alone would show the homogeneity of $\mathbb{R}^n$.

Ex. $(\mathbb{H}^2, g)$ hyp. plane

$$\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \Big/ \{\pm \mathrm{Id}\}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \dfrac{az+b}{cz+d}$ if we replace $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, we get the same

This is transitive: for this it suffices to show that $i \in \mathbb{H}^2$ can go anywhere (and anything can go to i)

For $\lambda \in \mathbb{R}$: $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} z = \lambda^2 z \Rightarrow$ can go anywhere on $i\mathbb{R}$ (move vertically)

$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} z = z + x \Rightarrow$ move horizontally

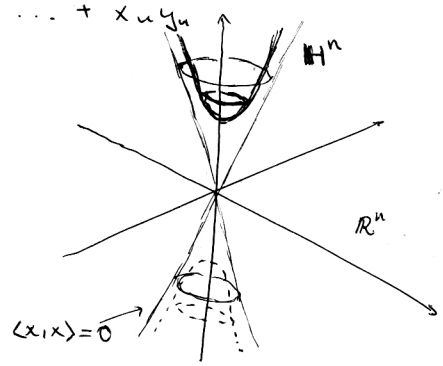
} move anywhere

Ex. $\mathbb{H}^n$ = hyperbolic n-space

Def. $\langle \cdot, \cdot \rangle_{1,n} : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$

$$(x, y) \longmapsto -x_0 y_0 + x_1 y_1 + \dots + x_n y_n$$

Def. $\mathbb{H}^n := \{ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1, \ x_0 > 0 \}$



$\langle x, x \rangle = 0 \iff x_0^2 = \| (x_1, \dots, x_n) \|^2$

$\forall p \in \mathbb{H}^n$: $\langle \cdot, \cdot \rangle_{1,n} \big|_{T_p \mathbb{H}^n}$ defines an inner product

$\Rightarrow$ we have a R. metric g on $\mathbb{H}^n$

Def. $(\mathbb{H}^n, g)$ is called the **hyperbolic n-space**.

$$\mathrm{Isom}^+(\mathbb{H}^n) = \mathrm{SO}(1,n)^0 = \{ A \in \mathrm{GL}_{n+1}(\mathbb{R}) \mid \langle Av, Aw \rangle_{1,n} = \langle v, w \rangle_{1,n}, \ A\mathbb{H}^n = \mathbb{H}^n \}$$

$-\mathrm{Id} \notin \mathrm{SO}(1,n)^0$

This acts transitively (PO), $\mathbb{H}^n$ is homogeneous.

Ex. Products of homog. spaces are homog. (Cartesian)

Def. X top space, $\varphi: X \to X$

1) φ is *involutive* (or an *involution*) if $\varphi^2 = id$

2) $p \in X$ is a *fixed point* of φ if $\varphi(p) = p$

3) a fixed point p of φ is called *isolated* if $\exists U \subseteq X$ open s.t. $p \in U$ and $\forall u \in U: \varphi(u) = u \Rightarrow u = p$.

Def. A Riemannian mf. M is *symmetric* if

- M is homogeneous, connected, complete;
- $\forall p \in M \ \exists \varphi: M \to M$ involutive isometry of which p is an isolated fixed pt.

Rmk. By homogeneity, it is equivalent to require M to have one isometry w/ an isolated fixed point.

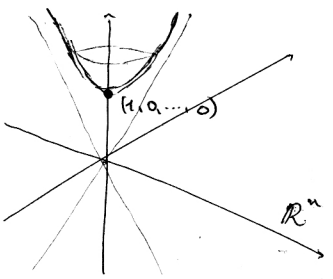
Ex. 1) S^n: $(x_1, \dots, x_{n+1}) \mapsto (-x_1, -x_2, \dots, -x_n, +x_{n+1})$

the fixed points are $(0, \dots, 0, \pm 1)$ $\longrightarrow$ isolated

2) $\mathbb{R}^n$: $x \mapsto -x$, the only f.p. is 0

3) H^n: $(x_0, x_1, \dots, x_n) \mapsto (x_0, -x_1, \dots, -x_n)$

$(1, 0, \dots, 0)$ is the fixed pt.

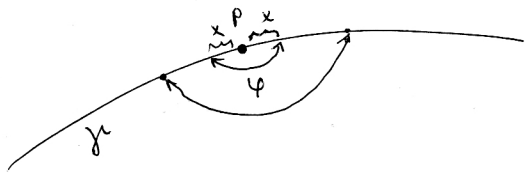


Locally the involution will always be of the form $x \mapsto -x$:

Prop. X Riem. mf., $\varphi: X \to X$ involutive isometry with isolated fixed pt. p. Then

1) $d\varphi|_p = -id$

2) For every geodesic $\gamma: \mathbb{R} \to X$ with $\gamma(0) = p$ we have $\varphi(\gamma(t)) = \gamma(-t) \ \forall t \in \mathbb{R}$



Pf: 1) $d(\varphi^2)|_p = d\varphi_{\varphi(p)} \circ d\varphi_p$ (chain rule)

$= d\varphi_p^2$ ($\varphi(p) = p$), where $d(\varphi^2) = id$

$\Rightarrow$ $d\varphi_p$ satisfies $x^2 - 1 = (x+1)(x-1) = 0$, every root has multiplicity 1

$\Rightarrow$ $d\varphi_p$ is diagonalisable and the possible eigenvalues are ± 1

NTS all ev. are -1. $\lightning$ Suppose not: $\exists v \in T_pX$, $d\varphi_p v = v$.

Let γ be the unique geodesic with $\gamma(0) = p$, $\gamma'(0) = v$

Consider the geodesic $\varphi \circ \gamma : \mathbb{R} \to X$ (φ is an isometry).

Since geodesics are uniquely determined by starting point and initial velocity, we get $\varphi \circ \gamma = \gamma$.

$\Rightarrow$ every pt. on γ is fixed under φ $\Rightarrow$ p is not isolated $\lightning$

$\Rightarrow d\varphi_p = -\mathrm{id}$, as desired

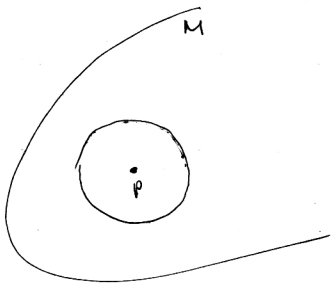
2) Basically the same: $\gamma(0) = p \Rightarrow d\varphi_p \dot\gamma(0) = \dot\gamma(0) \Rightarrow \varphi \circ \gamma(t) = \gamma(-t)$. □

Locally, such maps can always be found, but they need not extend to the whole mf.

Recall the exponential map. Let M be a Riem. mf.

$$\exp_p : T_pM (\cong \mathbb{R}^n) \longrightarrow M, \qquad \exp_p(tv) = \gamma(t)$$

where γ is the unique geodesic with $\gamma(0) = p$, $\dot\gamma(0) = v$



Moreover $\forall p \in M$ $\exists U \subseteq T_pM$ open set s.t.

$\exp_p : U \longrightarrow \exp_p(U)$ is a diffeomorphism.

By making U smaller, one can assume that

- U is symmetric, i.e. $U = -U$
- U is star-shaped, i.e. $\forall t \in [0,1]$: $tU \subseteq U$

Def. The <u>geodesic symmetry at p</u> is the map

$$\tau : \exp_p(U) \longrightarrow \exp_p(U)$$
$$\exp_p(v) \longmapsto \exp_p(-v)$$

Note that 2) in the Prop. shows that if φ is an involut. isom. $\Rightarrow$ $\Rightarrow$ φ agrees with τ near p.

Cor. A homogeneous space is symmetric iff it is connected, complete and $\forall p \in M$ the geodesic symmetry extends to a global isometry of M.

Note: usually our Riemannian mfs are nice, i.e. connected, complete, smooth, locally compact, ...

§2. Constructing symmetric spaces

Recall: topological group: group equipped with a topology s.t. all group operations are continuous.

Lemma. X connected homogeneous space,

$G := \mathrm{Isom}(X)^\circ$ = connected component of the identity in $\mathrm{Isom}(X)$

$K := \mathrm{Stab}_G(x)$ where $x \in X$ and Stab denotes the stabiliser

$\Rightarrow X \cong G/K$ (homeo.) and $K \subset G$ is compact.

PF: EXERCISES 1.2.1., 1.2.2. □

Recall: a Lie group: group equipped with a smooth manifold structure, all group operations are smooth.

Lemma. G Lie group, $K \subset G$ compact subgroup. Then there is a G-invariant Riemannian metric on G/K, and G/K with this metric is a homogeneous space.

Thm. (Haar, PO) H a locally compact Hausdorff group. $\Rightarrow$ there is a unique (up to scaling) σ-finite Borel measure μ on H s.t.

1) $\mu(C)$ is finite for $C \subseteq H$ compact
2) $\mu(hA) = \mu(A)$ $\forall A$ Borel $\forall h \in H$. □

Def: This μ is called the **Haar measure** on H.

Recall that σ-finiteness means that H is a countable union of sets with finite measure.

Note that $\mu(Ah) = \mu(A)$ need not be true.

Ex. 1) Finite groups: counting measure

2) $(\mathbb{R}^n, +)$: the Lebesgue measure is the unique translation invariant measure

PROOF OF LEMMA: Goal: left G-invariant Riemannian metric on G/K.

$\langle \cdot, \cdot \rangle : T_e G \times T_e G \longrightarrow \mathbb{R}$ any inner product ($e \in G$ is the identity elt)

Given $g \in G$ define $L_g : G \longrightarrow G$ by $L_g(h) := gh$ $\forall h \in G$.

Then we can define for $v, w \in T_g G$:

$$\langle v, w \rangle_g := \langle (dL_{g^{-1}})_g v, (dL_{g^{-1}})_g w \rangle$$

By construction this is left G-invariant. The problem is that it might not be right K-invariant.

$\forall g \in G$: $R_g : G \longrightarrow G$, $R_g(h) := hg$

$$(v, w)_g := \int_K \langle dR_k v, dR_k w \rangle_{gk} \, d\mu(k)$$

smooth function integrated over a compact space $\Rightarrow$ finite integral

$$(dR_k v, dR_k w)_{gk} \overset{\text{def}}{=} \int_K \langle dR_\ell \, dR_k v, dR_\ell \, dR_k w \rangle_{gk\ell} \, d\mu(\ell)$$

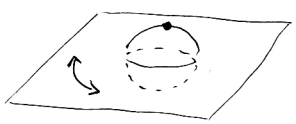
$$= \int_K \langle dR_{k\ell} v, dR_{k\ell} w \rangle_{gk\ell} \, d\mu(\ell)$$ chain rule

$$= \int_K \langle dR_{k\ell} v, dR_{k\ell} w \rangle_{gk\ell} \, d\mu(k\ell)$$ Haar measure

$$= \int_K \langle dR_{k'} v, dR_{k'} w \rangle_{gk'} \, d\mu(k') = (v, w)_g$$

$\Rightarrow$ right K-invariance $\Rightarrow$ descends to G/K.

Ex. 1) $S^n = SO(n+1)/SO(n)$

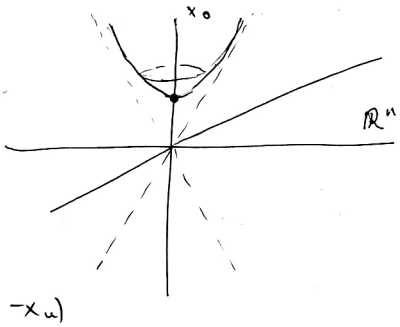


stabilising the North pole $\Leftrightarrow$ rotating the hyperplane $\longrightarrow SO(n)$

2) $\mathbb{R}^n = SO(n) \ltimes \mathbb{R}^n / SO(n)$ involution: $x \longmapsto -x$ with fixed pt $(0, \dots, 0)$

3) $H^n = SO(1,n)^0 / SO(n)$

stabiliser of $(1, 0, \dots, 0)$



involution: $(x_0, x_1, \dots, x_n) \longmapsto (x_0, -x_1, \dots, -x_n)$

17.4.2018

Prop. Let G be a connected linear Lie group,
K a compact subgroup,
σ an involutive automorphism of G
such that K is an open subgroup of $\{g \in G \mid \sigma(g) = g\} = C_G(\sigma)$.

Then G/K can be given the structure of a symmetric space s.t.

$$\tau(gK) := \sigma(g)\,K$$

is an involutive isometry of which eK is an isolated fixed pt.

For involutions, stabilisers and centralisers are the same.

PF: We already know that G/K is homogeneous.
We only need to check that τ has the desired properties.

σ is an involution $\Rightarrow$ τ is an involution

$G^+ := \underbrace{\langle \sigma \rangle}_{\simeq \mathbb{Z}/2\mathbb{Z}} \ltimes G$ (the action of σ on G comes from σ being an aut. of G)

$K^+ := \langle \sigma \rangle \times K$

Thus we get $G/K = G^+/K^+$.

The same construction as last week gives that τ is an isometry.

$\tau(eK) = \sigma(e)\,K = eK$ is trivial.

The last thing to check is that eK is isolated.

Suppose that $\exists gK$ fixed pt of τ s.t. g is close to e.

$gK = \tau(gK) = \sigma(g)K \Rightarrow \sigma(g) = g \cdot k$ for some $k \in K$

$g = \sigma^2(g) = \sigma(gk) = \sigma(g)\underbrace{\sigma(k)}_{k} = gk^2 \Rightarrow k^2 = e$

$g^{-1}\sigma(g) = k$, g is close to the identity e, $(\,)^{-1}$ and σ are continuous maps $\Rightarrow$ k is close to the identity e

EXERCISE: Show $\exists U$ open nbh. of $e \in G$ (lin. Lie group) s.t.
if u ell satisfies $u^2 = e$ then $u = e$.

Thus we get $k = e$, $\sigma(g) = g$, i.e. $g \in C_G(\sigma)$.

$K \subseteq C_G(\sigma)$ is open $\Rightarrow$ K contains an open nbh. of e $\Rightarrow$ contains e
$\Rightarrow$ $gK = eK$

□

Ex. 1) $S^n = SO(n+1)/SO(n)$ $\quad \sigma = \mathrm{diag}(-1, \dots, -1, 1)$

$K = C_G(\sigma) = \{ g \in G \mid \sigma L_g = L_g \sigma \}$

2) $\mathbb{R}^n = SO(n) \ltimes \mathbb{R}^n / SO(n)$ $\quad \sigma(k, v) = (k, -v)$

($SO(n)$ ↑, $\mathbb{R}^n$ ↑)

$K = SO(n)$ (which is clearly an open subgroup of itself)

3) $H^n = SO(1,n)^0 / SO(n)$ $\quad \sigma = \mathrm{diag}(1, -1, \dots, -1)$

$K = C_G(\sigma)$

Ex. $G = SL(n, \mathbb{R})$

$K = SO(n)$

$\sigma(g) = (g^{-1})^T$ involutive group automorphism.

By the Prop., we obtain a symmetric space $X = G/K$

Note that for $n = 2$ we get $X = H^2$ but this does not hold for $n > 2$.

Claim. $X = \{ A \in SL(n, \mathbb{R}) \mid A^T = A$ (symmetric), all eigenvalues are > 0 (pos. def.)$\}$

$\Downarrow$

all eigenvalues are real

Define $\alpha : G \times X \longrightarrow X$ by $\alpha(g, x) = g x g^T$.

EXERCISE: check that this is a transitive action.

Hint: a matrix x is positive (i.e. all eigenvalues are ≥ 0) if $x = yy^T$ for some y.

$K = \mathrm{Stab}_G(\mathrm{Id}) = SO(n) = \{ A \in SL_n(\mathbb{R}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \ \forall\, v, w \in \mathbb{R}^n \}$

$\Updownarrow$

$A^T A = \mathrm{Id}$

The following part involves some theory of Lie groups; one should not worry too much about not understanding it in full detail.

To get a metric, we first identify the tangent space:

$T_e X = \{ A \in \mathrm{Mat}_{n \times n}(\mathbb{R}) \mid A$ symmetric, $\mathrm{tr}\, A = 0 \}$

$\downarrow$

comes from $1 = \det(\exp A) = \exp(\mathrm{tr}\, A)$

Define $\langle A \mid B \rangle = \mathrm{tr}(AB^T)$: will be an inner product.

For $A \in X$, let $\tau(A) := A^{-1}$. τ is an isometry.

Thus we obtain a symmetric space.

Def. 1) A symmetric space is <u>irreducible</u> if its universal cover is not isometric to a product of symmetric spaces.

(Ex.: S^n, H^n. Non-ex.: $\mathbb{R}^n$ for $n>1$.)

2) A Riemannian mf. is <u>flat</u> if its Riemann curvature tensor is zero at every point.

This is equivalent to every point having an open nbh. that is isometric to an open subset of $\mathbb{R}^n$.

(Ex. $\mathbb{R}^n$, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$)

3) A Lie group is <u>simple</u> if

- it is non-abelian
- it has no connected closed nontrivial normal subgroup.

(Ex. $SL_n(\mathbb{R})$ (PO))

4) G_1, G_2 Lie groups are <u>isogenous</u> if

- $\exists$ finite index $G_i' < G_i$ for $i=1,2$
- $\exists$ finite $N_i \triangleleft G_i$ for $i=1,2$

s.t. $G_1'/N_1 \simeq G_2'/N_2$.

5) G is <u>semisimple</u> if it is isogenous to a product of simple groups.

Prop G connected non-compact simple Lie group with finite center

Then G has a maximal compact subgroup K s.t.

- K is unique up to conjugation
- G/K is a simply connected irreducible symmetric space (1)
- G/K has non-positive sectional curvature everywhere and it is not flat. (2)

Every symmetric space with properties (1) and (2) is of this form.

Most of these statements should not come as a surprise in light of our previous work and the above definitions.

Def. A complete Riemannian mf. is <u>locally symmetric</u> if its universal cover is a symmetric space.

Ex. Symmetric $\Rightarrow$ locally symmetric.

$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is loc. symmetric

Rmk. Loc. symmetric $\Rightarrow$ the geodesic symmetry is a local isometry

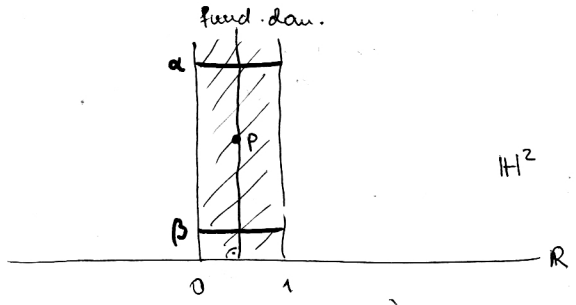
Recall. $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ with $ds^2 = \frac{dx^2 + dy^2}{y^2}$ being the metric. Geodesics: horizontal lines and half-circles.

$g : \mathbb{H}^2 \to \mathbb{H}^2$, $z \mapsto z+1$ $\qquad \operatorname{Isom}^+(\mathbb{H}^2, ds^2) = PSL_2(\mathbb{R})$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{az+b}{cz+d}$

$PSL_2(\mathbb{Z}) = \Gamma = \langle g \rangle \simeq \mathbb{Z}$

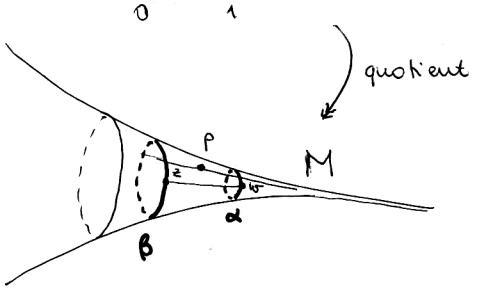
$\mathbb{H}^2/\Gamma$ is loc. symmetric

The quotient is homeomorphic to a cylinder.



Why is this not a symmetric space?

There is no homotopically nontrivial closed loop through z of length $< \varepsilon$ (for ε small enough)



On the other hand, there is such a loop through w.

$\to$ No isometry $M \to M$ can send z to w.

Also, a geodesic symmetry w to z flips the two ends.

Group version

Def. X loc cpt top space, $G \curvearrowright X$ by homeomorphisms. The action is called <u>properly discontinuous</u> if $\forall K \subset X$ cpt: $\{g \in G \mid gK \cap K\}$ is finite.

Ex. $\mathbb{Z} \curvearrowright \mathbb{R}$: $m \cdot x = x + m$ $(m \in \mathbb{Z},\ x \in \mathbb{R})$ $\to$ prop. discont.

$\mathbb{R} \curvearrowright \mathbb{R}$ $\quad y \cdot x = x + y$ $\quad y \in (0,1)$: $(y + [0,1]) \cap [0,1] \neq \emptyset$, it is even uncountable

$\to$ this is not prop. discont.

Equivalent def. (locally symmetric) M complete mf. is loc. symmetric if there is a symmetric space X and $\Gamma < \operatorname{Isom}(X)$ s.t.

- Γ acts prop. discontinuously and freely on X
- M is isometric to $\Gamma \backslash X$

So $M = \Gamma \backslash G/K$

In [WM], we move on now to Ch. IV.

From now on: G semisimple (linear) Lie group with finitely many components.

$\Gamma < G$ lattice (def. comes later)

Ex. $G = SL_n(\mathbb{R})$, $G = SO(m,n) = \{ A \in SL(m+n, \mathbb{R}) \mid \langle Ax, Ay \rangle_{m,n} = \langle x, y \rangle_{m,n} \ \forall x,y \}$

where $\langle x, y \rangle_{m,n} = -\sum_{i=1}^{m} x_i y_i + \sum_{i=m+1}^{n} x_i y_i$

Lemma. $\Lambda < G$ discrete subgroup $\Rightarrow \exists$ a strict fundamental domain for $\Lambda \curvearrowright G$, i.e. $\exists \mathcal{F} \subset G$ s.t. $\mathcal{F}$ is Borel and $\mathcal{F} \to G/\Lambda$ is bijective.

(So the strictness means that we require injectivity.)

Pf: Λ discrete and non-empty (since it is a group) $\Rightarrow \exists U \subset G$ open s.t. $U^{-1}U \cap \Lambda = \{e\}$ (EXERCISE)

G is second-countable $\longrightarrow \exists g_n \in G$ s.t. $\bigcup_{n=1}^{\infty} g_n U = G$

$$\mathcal{F} := \bigcup_{n=1}^{\infty} \left(g_n U \setminus \{ \bigcup_{i<n} g_i U \cap \Lambda \} \right)$$

This is clearly Borel and a strict fund. domain. □

Prop: $\Lambda < G$ discrete subgp., μ Haar measure on G. $\Rightarrow \exists!$ (up to scalar) σ-finite, G-invariant Borel measure ν on G/Λ:

1) $A \subset G$ Borel with $A\Lambda = A \Rightarrow \nu(A/\Lambda) = \mu(A \cap \mathcal{F})$.

(Every Borel subset of G/Λ is of the form A/Λ for such an A.)

2) $A \subset G$ Borel $\Rightarrow \mu(A) = \int_{G/\Lambda} \#(A \cap x\Lambda) \, d\nu(x\Lambda)$

Pf: left-invariant measure is also right-invariant, this follows from semisimplicity Ex. 4.1.7., 4.1.8. □

Convention. From now on μ and ν satisfy (1) and (2), i.e. the scaling between them is fixed.

Cor. $A \subset G$ Borel, $\varphi: G \twoheadrightarrow G/\Lambda$ quotient map $\Rightarrow \nu(\varphi(A)) = \mu(A)$. □

Rmk. These μ and ν are really volumes since μ comes from a volume form on G.

Def $\Gamma < G$ is a **lattice** if: Γ is discrete and G/Γ has finite volume.

Ex $SL_2(\mathbb{Z}) < SL_2(\mathbb{R})$

Prop $\Lambda < G$ discrete subgroup, μ Haar measure on G. TFAE:

1) Λ is a lattice
2) $\exists\, \mathcal{F}$ strict fund. domain for $\Lambda \curvearrowright G$ with $\mu(\mathcal{F}) < \infty$
3) $\exists\, C \subseteq G$ Borel set s.t. $C\Lambda = G$ and $\mu(C) < \infty$

PF: 1) $\Rightarrow$ 2) Use the previous Prop..

$$\infty \overset{1)}{>} \nu\left(G/\Lambda\right) = \mu(G \cap \mathcal{F}) = \mu(\mathcal{F})$$

2) $\Rightarrow$ 3) $C = \mathcal{F}$ satisfies 3)

3) $\Rightarrow$ 1) $\nu(G/\Lambda) = \int_{G/\Lambda} 1 \, d\nu(x\Lambda) \leq \int_{G/\Lambda} \underbrace{\#(C \cap x\Lambda)}_{\geq 1} \, d\nu(x\Lambda) = \mu(C) \overset{3)}{<} \infty$

Def $\Lambda < G$ closed subgroup is called **cocompact** (or **uniform**) if G/Λ is compact.

Ex $\mathbb{Z}^n < \mathbb{R}^n$, $\{e\} < SO(n)$ (since $SO(n)$ is compact)

Non-ex $SL_2(\mathbb{Z}) < SL_2(\mathbb{R})$

Note that cocompactness $\not\Rightarrow$ finite volume.

Cor Every cocompact discrete subgroup of G is a lattice.

Every finite index subgroup of a lattice is a lattice.

PF: Ex. 4.1.12., 4.1.13.

Commensurability

Def Hgp., $\Lambda_1, \Lambda_2 < H$ are commeasurable if $\Lambda_1 \cap \Lambda_2$ is finite index in both Λ_1 and Λ_2.

Ex $a\mathbb{Z}, b\mathbb{Z} < \mathbb{R}$ are comm. iff $\frac{a}{b} \in \mathbb{Q}$.

Recall M mf., $M = \tilde{M}/\pi_1(M)$ where $\tilde{M}$ is the univ. cover.

$$\left\{\begin{array}{l}\text{covers of } M \text{ up to} \\ \text{base pt preserving iso}\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\text{subgroups of} \\ \pi_1(M)\end{array}\right\}$$

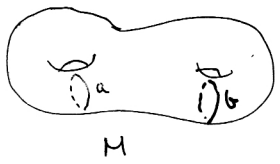
degree — index

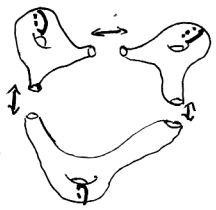
(look it up in Hatcher)

Two given covers $M_1 \to M \leftarrow M_2$ correspond to $\Lambda_1, \Lambda_2 < \pi_1(M)$.

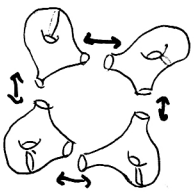
Then Λ_1 and Λ_2 are commensurable iff M_1, M_2 have a common finite cover.

Ex.





$\Lambda_1 \lhd \pi_1(M)$ (index 3) $\xrightarrow{\text{alg. intersect. with } a} \mathbb{Z} \xrightarrow{\text{mod } 3} \mathbb{Z}/3\mathbb{Z}$



$\Lambda_2 = \ker\left(\begin{matrix}\text{algebraic intersection} \\ \text{with } b \text{ mod } 4\end{matrix}\right)$

$\pi_1(M) \longrightarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$

What is the common finite cover?

Def $g \in G$ is said to <u>commensurate</u> $\Gamma < G$ if $g\Gamma g^{-1}$ is commens. to Γ. 24.04.2018

<u>Commensurator</u> of Γ: $\underline{\mathrm{Comm}_G(\Gamma)} := \{ g \in G \mid g \text{ commensurates } \Gamma \}$

Note that $N_G(\Gamma) = \{ g \in G \mid g\Gamma g^{-1} = \Gamma \} < \mathrm{Comm}_G(\Gamma)$, but usually the normalisator is a lot smaller. (In this notation, $<$ probably allows $=$.)

Ex. $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$ (we will prove that Γ is a lattice later)

Then $N_\Gamma(G)$ is commensurable to Γ and $\mathrm{Comm}_G(\Gamma) > SL(n, \mathbb{Q})$ is dense in G.

Def. Λ_1, Λ_2 are <u>abstractly commensurable</u> if $\exists\, \Lambda_i' < \Lambda_i$ finite index subgroups s.t. $\Lambda_1' \cong \Lambda_2'$.

Irreducible lattices

Def. Γ is called <u>irreducible</u> if (1) for every non-compact closed normal subgp. $N \lhd G^\circ$ the set ΓN is dense

and (2) Γ is infinite.

Ex. G simple $\Rightarrow$ every (infinite) lattice is irreducible.

Counterex. $G = G_1 \times G_2$, $\Gamma = \Gamma_1 \times \Gamma_2$ where $\Gamma_i < G_i$ lattice.

Then $N = G_1^\circ$ does not satisfy (1).

Prop. Assume that • G has trivial center and

• Γ projects densely in the maximal component factor of G.

Then $\exists$ a decomposition $G = G_1 \times \dots \times G_r$ and lattices $\Gamma_i < G_i$ s.t.

• Γ is commensurable to $\Gamma_1 \times \dots \times \Gamma_r$ and

• $\Gamma_i < G_i$ is irreducible.

Proof later, needs Borel density.

Def: A locally symmetric space $\Gamma\backslash X$ is called irreducible if there are no nontrivial locally symmetric spaces $\Gamma_1\backslash X_1$ and $\Gamma_2\backslash X_2$ s.t. $\Gamma_1\backslash X_1 \times \Gamma_2\backslash X_2$ is a finite cover of $\Gamma\backslash X$.

Prop. There are locally symmetric spaces $\Gamma_1\backslash X_1, \dots, \Gamma_r\backslash X_r$ s.t.

• $\Gamma_i\backslash X_i$ are irreducible

• $\Gamma_1\backslash X_1 \times \dots \times \Gamma_r\backslash X_r$ is a fin. cover of $\Gamma\backslash X$.

Pf: Induction. □

Prop. M irreducible loc symm space s.t. the univ. cover X of M has no compact factors and no flat factors and for any nontrivial product decomposition $X = X_1 \times X_2$ the img of X_i is dense in M.

Pf: EXERCISE. □

We wish to understand the structure of a loc. symm. space from its lattice.

Lemma: $\Gamma < G$ discrete subgp, $K < G$ cpt. subgp.

Then $\Gamma\backslash G$ is cpt. $\iff$ $\Gamma\backslash G/K$ is cpt. and

$\Gamma\backslash G$ has finite volume $\iff$ $\Gamma\backslash G/K$ has finite volume.

Pf: $\Gamma\backslash G$ cpt $\Rightarrow$ $\Gamma\backslash G/K$ cpt. since the quotient operation preserves compactness in general.

Let $\Gamma \backslash G/K$ be cpt., $(g_n)_n$ be a sequence in $\Gamma \backslash G$. Want to find a convergent subseq.

$[g_n]_n$: the projection to $\Gamma \backslash G/K$ $\Rightarrow$ has a convergent subseq. $[g_{n_j}]_j$

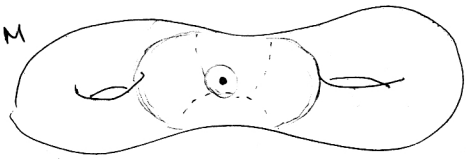
Pick representatives $\tilde{g}_{n_j}$ in each class, $g_{n_j} = \tilde{g}_{n_j} \cdot k_{n_j}$ $\forall j$

$k_{n_j} \in K$ $\rightarrow$ $\exists$ convergent subsequence $\left(k_{n_{j_\ell}}\right)_\ell$

$\Rightarrow \left(g_{n_{j_\ell}}\right)_\ell = \left(\tilde{g}_{n_{j_\ell}} \cdot k_{n_{j_\ell}}\right)_\ell$ is convergent in $\Gamma \backslash G$. ✓

The finite volume equivalence is EXERCISE 1.3.6. □

Def. M Riemannian mf., $x \in M$



Injectivity radius at x:

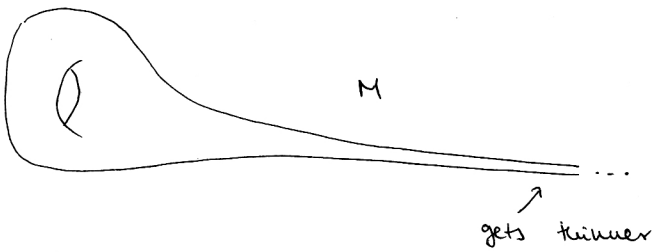
$$\operatorname{inj}_x(M) := \sup\left\{ r \geq 0 \;\middle|\; \exp_x : \underbrace{\left(\exp_x^{-1}(B_r(x))\right)^\circ}_{\text{connected component containing } 0.} \longrightarrow B_r(x) \text{ is a diffeomorphism} \right\}$$

I.e. the inj radius is "the largest radius s.t. the ball does not overlap with itself." (i.e. $\exp_x$ is a local diffeo.) Here $B_r(x)$ denotes the open ball with radius r.

Def. **Injectivity radius of M:** $\operatorname{inj}(M) := \inf\{ \operatorname{inj}_x(M) \mid x \in M \}$

Observation: if M is compact then $\operatorname{inj}(M) > 0$. This need not hold for non-compact manifolds M.

Ex.



$\Rightarrow \operatorname{inj}(M) = 0.$

$\mathbb{H}^2/SL_2(\mathbb{Z})$ is like this (although it is not a mf., but the def. can be extended to orbifolds)

Prop. $g \in G$ defines a map $\varphi_g : G \longrightarrow G/\Gamma$, $x \longmapsto xg\Gamma$

Then G/Γ compact $\Leftrightarrow$ $\exists U$ open nbhd. of e s.t. $\forall g \in G$: $\varphi_g : U \to G/\Gamma$ is injective.

Pf: "$\Rightarrow$" Let $\varphi : G \longrightarrow G/\Gamma$, $x \longmapsto x\Gamma$, i.e. $\varphi = \varphi_e$.

This is a covering map (follows from the discreteness of Γ and structure of G)

i.e. $\forall p \in G/\Gamma$ $\exists V_p$ connected open neighbourhood s.t. restricting φ to a connected component of $\varphi^{-1}(V_p)$ yields a diffeomorphism.

<u>Claim.</u> $\exists$ connected nbh. U of e in G s.t. $\forall p \in G/\Gamma$ $\exists p' \in G/\Gamma$ s.t. $U \cdot p \subseteq V_{p'}$.

Note that Claim $\Rightarrow$ Prop.

PF OF CLAIM: V_p open $\Rightarrow$ $\exists U_p$ nbh. of $e \in G$ s.t. $U_p \, p \subset V_p$.

Define new open nbhs of $e \in G$ W_p s.t. $v, w \in W_p \Rightarrow v \cdot w \in U_p$.

$\{W_p \cdot p \mid p \in G/\Gamma\}$ is an open covering of G/Γ $\xrightarrow{\text{compactness}}$ $\{W_{p_i} p_i\}_{i=1}^N$ finite subcover

$U := \bigcap_{i=1}^N W_{p_i}$ open nbhd of e in G.

$q \in W_{p_i} p_i \Rightarrow uq \in U_{p_i} p_i \subset V_{p_i}$ $\forall u \in U$ by construction. ✓

"$\Leftarrow$" We will prove that if $\emptyset \neq U \subset G$ open precompact and G/Γ not cpt. $\Rightarrow \exists g \in G$ s.t. $\varphi_g : U \longrightarrow G/\Gamma$ is not injective.

Let $C \subset G$ be compact.

$\underbrace{\left(G/\Gamma\right)}_{\text{cpt.}} \setminus \underbrace{U^{-1} G}_{\text{non-cpt.}} \neq \emptyset$

We can thus build a sequence $(g_n)_n$ in G such that $g_n U \subset G/\Gamma$ are all disjoint:

Take $g_0 = e$, $C_i := \overline{g_i U} \cup C_{i-1}$, $g_i \in \left(G/\Gamma\right) \setminus \left(U^{-1} C_i\right)$

Γ is a lattice $\Rightarrow$ $\mathrm{vol}\left(G/\Gamma\right) < \infty$

$\Rightarrow \mathrm{vol}\left(g_n U\right) \longrightarrow 0$ as $n \to \infty$

$\Rightarrow \exists n$: $\varphi_{g_n} : U \longrightarrow G/\Gamma$ is not injective (otherwise the volume would be preserved). □

This was still kind of geometric.

We want an <u>algebraic version</u>.

<u>Notation.</u> H group, $a, b \in H$, $A, B \subset H$.

$${}^b a := bab^{-1}, \qquad {}^B a := \{{}^b a \mid b \in B\}$$

$${}^b A := \{{}^b a \mid a \in A\}, \qquad {}^B A := \{{}^b a \mid a \in A, b \in B\}$$

Corollary. G/Γ is cpt. $\iff$ $e \in G$ is not an accumulation point of Γ.

Pf: $\varphi_g|_U$ injective $\iff \exists u_1, u_2 \in U$ s.t. $u_1 g\Gamma = u_2 g\Gamma$

$\iff g\Gamma \cap UU^{-1} = \{e\}$ □.

26.04.2018

Simple Lie groups (intermezzo)

Classical Lie groups: all simple

1) $SL(n,\mathbb{R})$ $n \geq 2$ special linear group

2) $SO(m,n)^0 = \{ g \in SL(m+n,\mathbb{R}) \mid g^T I_{m,n} g = I_{m,n} \}^0$ where $I_{m,n} = \begin{pmatrix} -1 & & & & \\ & \ddots & & & 0 \\ & & -1 & & \\ & & & +1 & \\ 0 & & & & \ddots \\ & & & & & +1 \end{pmatrix}$ (m entries -1, n entries $+1$),

↑ preserve bilinear forms

$m+n \geq 3$ special orthogonal group

3) $SU(m,n) = \{ g \in SL(m+n,\mathbb{C}) \mid g^* I_{m,n} g = I_{m,n} \}$ where $g^* = \overline{g}^T$ is the adjoint

↑ preserve sesquilinear forms

$m+n \geq 3$ special unitary group

4) $\mathcal{I}_{2m} := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$ $Sp(2m,\mathbb{R}) = \{ g \in SL(2m,\mathbb{R}) \mid g^T \mathcal{I}_{2m} g = \mathcal{I}_{2m} \}$

↑ preserve antisymmetric forms

$m \geq 1$ symplectic group

Different fields:

1) $SL(n,\mathbb{C})$, $SL(n,\mathbb{H})$ where $\mathbb{H}$ = quaternions $= \{ a_0 + a_1 i + a_2 j + a_3 k \mid a_n \in \mathbb{R} \}$

with $ij = -ji = k$, $i^2 = j^2 = k^2 = -1$

$\mathbb{H} \cong \mathbb{C}^2$ as a vector space. Thus taking det in $Mat_{n\times n}(\mathbb{H})$ can be done by taking it in $Mat_{2n\times 2n}(\mathbb{C})$. (This det is called the reduced norm.)

2) $SO(n,\mathbb{C}) = \{ g \in SL(n,\mathbb{C}) \mid g^T g = I_n \}$

$SO(n,\mathbb{H}) = \{ g \in SL(n,\mathbb{H}) \mid tr(g^T)\, g = I_n \}$ where $tr(a_0 + a_1 i + a_2 j + a_3 k) = a_0 + a_1 i - a_2 j + a_3 k$

3) $Sp(2m,\mathbb{C}) = \{ g \in SL(2m,\mathbb{C}) \mid g^T \mathcal{I}_{2m} g = \mathcal{I}_{2m} \}$ complex symplectic group

4) $Sp(m,n) = \{ g \in SL(m+n,\mathbb{H}) \mid g^* I_{m,n} g = I_{m,n} \}$

Thm. (E. Cartan) Every simple Lie group is isogenous either to a classical group or one of the finitely many exceptional groups.

From now on, think of a simple Lie group as one of these.

Def. A matrix $u \in SL(n, \mathbb{C})$ is called <u>unipotent</u> if $\exists m \in \mathbb{N}: (u - I_n)^m = 0$.

Equivalently, all eigenvalues are 1.

Note that any (semi)simple Lie group is in $SL(n, \mathbb{R})$ for large enough n.

Ex. $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{C})$: $(u - I_2)^2 = 0$. unipotent

Cor. If Γ has a nontrivial unipotent element, then G/Γ is not compact.

Pf: (1) $u := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$. Let $a := \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \in SL(2, \mathbb{R})$.

$a^n u a^{-n} = \begin{pmatrix} 1 & 2^{-2n} \\ 0 & 1 \end{pmatrix} \xrightarrow{n \to \infty} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. hence e is an accumulation point of $G\Gamma$. $\Rightarrow$ G/Γ non-cpt.

We now show that (1) can be generalised.

<u>Jacobson–Morosov lemma</u>: There is a continuous homomorphism (PO)

$\varphi: SL(2, \mathbb{R}) \longrightarrow G$ with $\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = u$ where G is non-compact, $u \in G$.

Set $g_n := \varphi\left(a^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} a^{-n}\right) \xrightarrow{n \to \infty} e \in G$. $\Rightarrow$ G/Γ non-cpt.

Note. Knapp: Structure theory of semisimple Lie groups $\longrightarrow$ quick overview

Fulton–Harris: (sth with "introduction" and "rep. theory")

Prop Λ lattice in H, $C \subset H$. Then the img of C in H/Λ is precompact iff $e \in H$ is not an accumulation pt of $C\Lambda$.

Pf: EXERCISE 4.4.2., just put the above proofs together.

Prop. (Mahler Compactness Criterion)

$C \subseteq SL(\ell, \mathbb{R})$. Then the img of C in $SL(\ell, \mathbb{R})/SL(\ell, \mathbb{Z})$ is precompact iff 0 is not an accumulation point of $C\mathbb{Z}^\ell = \{cv \mid c \in C, v \in \mathbb{Z}^\ell\} \subseteq \mathbb{R}^\ell$.

Pf: "$\Rightarrow$" $\pi(C)$ precompact $\Rightarrow$ $\exists\, C_0 \subseteq SL(\ell, \mathbb{R})$ compact s.t. $C \subseteq C_0 SL(n, \mathbb{Z})$ (Ex. 4.1.11.)

where $\pi: SL(\ell, \mathbb{R}) \twoheadrightarrow SL(\ell, \mathbb{R})/SL(\ell, \mathbb{Z})$ is the quotient map.

By enlarging C wma $C = C_0 SL(n, \mathbb{Z})$.

$\Rightarrow$ $C(\mathbb{Z}^\ell \setminus \{0\}) = C_0 SL(n, \mathbb{Z})(\mathbb{Z}^\ell \setminus \{0\}) = C_0(\mathbb{Z}^\ell \setminus \{0\})$

C_0 cpt, hence closed. $\mathbb{Z}^\ell \setminus \{0\}$ closed $\Rightarrow$ $C_0(\mathbb{Z}^\ell \setminus \{0\})$ closed, hence contains all its accumulation pts.

$\lightning$ If $0 \in C(\mathbb{Z}^l \setminus \{0\})$ then $\exists c \in C$, $v \in \mathbb{Z}^l \setminus \{0\}$ s.t. $cv = 0$. But elts of C are invertible. $\lightning$

"$\Leftarrow$" Suppose $(g_n)_n$ is a sequence of elts in $SL(n, \mathbb{R})$ s.t. 0 is not an acc. point of $\bigcup_{n \geq 0} g_n \mathbb{Z}^l$.

Our goal is to show that $\exists (\gamma_n)_n$ in $SL(n, \mathbb{Z})$ s.t. $(g_n \gamma_n)_n$ has a convergent subsequence.

For each n define

- $v_n^1 \in \mathbb{Z}^l \setminus \{0\}$ s.t. $\| g_n v_n^1 \|$ has minimal norm
- $v_n^i \in \mathbb{Z}^l \setminus \mathrm{Span}_{\mathbb{R}} \{ g_n v_n^1, \dots, g_n v_n^{i-1} \}$ s.t. $\left\| \mathrm{proj}_{\{g_n v_n^1, \dots, g_n v_n^{i-1}\}^\perp}(g_n v_n^i) \right\|$ is minimal

$K_n := \mathrm{conv}\{v_n^1, \dots, v_n^l\}$

(Note that $v_n^1, \dots, v_n^{i-1}$ are lin. independent.)

$K_n \cap (\mathbb{Z}^l \setminus \{0\}) = \{v_n^1, \dots, v_n^l\}$ by norm-minimality (*)

Define $\gamma_n := [v_n^1 \ \dots \ v_n^l] \in \mathrm{Mat}_{l \times l}(\mathbb{Z})$.

Claim. $\det(\gamma_n) = \pm 1$.

Pf: $\det(\gamma_n) = \pm \mathrm{vol}(Q)$ where $Q = \mathrm{conv}\left\{ \sum_{i=1}^l \varepsilon_i v_n^i \;\middle|\; \forall \varepsilon_i \in [0,1] \right\}$

We claim that Q is a fund domain for $\mathbb{Z}^l \curvearrowright \mathbb{R}^l$.

This would imply $\mathrm{vol}(Q) = \mathrm{vol}\left(\mathbb{R}^l / \mathbb{Z}^l\right) = 1$, hence $\det(\gamma_n) = \pm 1$.

$\mathbb{Z}^l Q = \mathbb{R}^l$ clearly holds.

NTS: $(v + \mathring{Q}) \cap \mathring{Q} = \emptyset$ $\forall v \in \mathbb{Z}^l$. This follows from (*).

(If they would intersect, we would obtain a nontriv lin. combination.) □

Note: e_j standard basis vectors for $\mathbb{R}^l$. Then $\gamma_n e_j = v_n^j$

Since our proof is only up to $SL(l, \mathbb{Z})$, we may replace g_n by $g_n \gamma_n$.

$\Rightarrow$ we have $v_n^j = e_j$.

$$\prod_{i=2}^{\ell} \left\| \operatorname{proj}_{\{g_u v_u^1, \dots, g_u v_u^{i-1}\}} v_u^i \right\| = \operatorname{vol}(Q) = 1 \qquad (**)$$

Goal: show that the sequence $g_u v_u^i$ is bounded.

Claim: $\left\| \operatorname{proj}_{\{g_u v_u^1, \dots, g_u v_u^{i-1}\}^\perp} g_u v_u^i \right\| \geq \left\| \operatorname{proj}_{\{g_u v_u^1, \dots, g_u v_u^{i-2}\}^\perp} g_u v_u^{i-1} \right\| \cdot \frac{1}{2}$

We can replace v_u^2 by $v_u^2 + k v_u^1$ ($k \in \mathbb{Z}$) in the above construction; all properties remain satisfied.

We do an almost Gram–Schmidt:

we want to replace v_u^2 by $v_u^2 - \frac{\langle v_u^1, v_u^2 \rangle}{\langle v_u^1, v_u^1 \rangle} \cdot v_u^1$, but this won't work since the fraction may not be in $\mathbb{Z}$.

Instead: we take the closest integer

$$\Rightarrow \left\| \operatorname{proj}_{v_u^1} g_u v_u^2 \right\| \leq \frac{1}{2} \left\| g_u v_u^1 \right\|$$

$$\Rightarrow \left\| g_u v_u^1 \right\| \leq \left\| g_u v_u^2 \right\| = \left\| \operatorname{proj}_{v_u^1 \perp} g_u v_u^2 \right\| + \left\| \operatorname{proj}_{v_u^1} g_u v_u^2 \right\|$$

$$\uparrow \text{ by minimality} \leq \left\| \operatorname{proj}_{v_u^1 \perp} g_u v_u^2 \right\| + \frac{1}{2} \left\| g_u v_u^1 \right\|$$

Same for higher indices.

Combine this with $(**)$ $\Rightarrow \|g_u v_u^i\|$ is bounded $\Rightarrow$ the matrix entries $g_u \gamma_u$ are bounded

Note that the above proof does not use the yet to be proven fact that $SL(\ell, \mathbb{Z})$ is a lattice.

We continue working towards the Borel Density Theorem.

The Borel Density Theorem

Thm. (Borel, but not the BDT)

Assume that G has no compact factors, G is connected;

V is a finite dimensional $\mathbb{R}$- or $\mathbb{C}$-vector space;

$\rho: G \longrightarrow GL(V)$ is a cont. homomorphism.

1) Then every $\rho(\Gamma)$-invariant vector in G is $\rho(G)$-invariant.

2) Then every $\rho(\Gamma)$-invariant sub-vector space $W \subseteq V$ is $\rho(G)$-invariant.

(i.e. if the lattice does not do anything interesting, neither can the group.)

Pf: Application of ergodic theory: combination of probability measures and group theory. $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Note that 2) $\Rightarrow$ 1):

Suppose Γ fixes v $\Rightarrow$ Γ fixes $\mathbb{K}v \overset{2)}{\Rightarrow} G$ fixes $\mathbb{K}v$

Thus we get a homomorphism $G \longrightarrow \mathbb{K}^\times$.

Claim: this is trivial.

$N := \ker(G \longrightarrow \mathbb{K}^\times) \subseteq G$ closed normal subgroup

Semisimplicity $\Rightarrow$ $N = G$ or $N = \{e\}$. The latter cannot happen since then $G \simeq \mathbb{K}^\times$, but semisimple groups are non-abelian.

$\Rightarrow N = G$, i.e. $G \longrightarrow \mathbb{K}^\times$ is trivial, G fixes v.

Lemma. (Poincaré Recurrence Theorem)

Let (X, d) be a separable metric space, $T: X \longrightarrow X$ (translation), μ a T-invariant Borel measure, $\mu(X) < \infty$.

Then for μ-a.e. $x \in X$ $\exists\, n_k \to \infty$ sequence s.t. $T^{n_k} x \to x$.

Pf: $A_\varepsilon := \{a \in X \mid \forall m > 0 \quad d(T^m a, a) > \varepsilon\}$ "bad" points

STS: $\forall \varepsilon > 0 \quad \mu(A_\varepsilon) = 0$

↯ Suppose not. Claim: $\exists B \subseteq A_\varepsilon$ with $\mu(B) > 0$, $\operatorname{diam}(B) < \varepsilon$

PF OF CLAIM: Separability $\Rightarrow$ A_ε is the countable union of fin. many balls.

$$0 < \mu(A_\varepsilon) < \sum_{i=1}^{\infty} \mu(A_\varepsilon \cap B_\varepsilon(p_i)) \quad \Rightarrow \quad \exists p_i : \ \mu(A_\varepsilon \cap B_\varepsilon(p_i)) > 0$$

Set $B := A_\varepsilon \cap B_\varepsilon(p_i)$. This satisfies the conditions.

$$\mu(T^{-m} B) = \mu(B) \quad \forall m$$

$$\exists n > m : \quad T^{-m} B \cap T^{-n} B \neq \emptyset \quad \text{since} \quad \mu(X) < \infty$$

$$\overset{T^n}{\Rightarrow} \quad T^{n-m} B \cap B \neq \emptyset$$

$$\Rightarrow \quad \exists x \in B : \ d(T^m x, x) < \varepsilon.$$

PF OF 1): let v be a fixed vector of $\rho(\Gamma)$. Goal: $\rho(G)$ fixes v.

Non-ptness $\Rightarrow$ $\exists\, u \in G \setminus \{e\}$ unipotent elt

Claim: $N := \overline{\langle u \in G,\ u \text{ unipotent} \rangle} = G$.

PF: Fact: unipotency is conjugacy invariant.

$\Rightarrow$ N normal and closed $\Rightarrow$ we are done if G is simple.

If not: same for each factor.

New goal: show that every unipotent in G fixes v.

Since v is a $\rho(\Gamma)$-fixed vector, we get a map

$$\overline{\rho} : G/\Gamma \longrightarrow V$$

$$g\Gamma \longmapsto \rho(g) \cdot v$$

Thus if ν is the finite volume G-invariant measure on G/Γ, then $\overline{\nu} := \rho_* \nu$ is a finite volume measure on V.

Poincaré recurrence $\Rightarrow$ if $u \in G$ unipotent then $\left(\rho(u^n g) v\right)_{n \in \mathbb{N}}$ has a convergent subsequence for a.e. $g \in G$.

Claim: The coordinates of $\rho(u^n g) v$ are polynomial functions of n.

Claim $\Rightarrow$ 1): the only polynomial having a convergent subseq of values is a constant. $\Rightarrow$ $\rho(u^n g) v$ constant for a.e. g

By continuity, $\rho(u^n)v$ is constant $\Rightarrow \rho(g)v = v \quad \forall g \in G$.

PF OF CLAIM: This is where we will use unipotency.

Claim'. $\rho(u)$ is unipotent.

Claim' $\Rightarrow$ Claim: $u \in GL(V)$ unip. $\Leftrightarrow \exists N \in Mat(V),\ k \in \mathbb{N}: u = Id + N,\ N^k = 0$.

$$u^n v = (Id+N)^n v = \sum_{l=0}^{n} \binom{n}{l} N^l \cdot v = \sum_{l=0}^{k} \binom{n}{l} N^l v, \text{ which is polynomial in } n.$$

PF OF CLAIM': Recall that $\exists a_u \in SL_2(\mathbb{R})$ s.t. $a_u \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} a_u^{-1} \longrightarrow Id$.

$$\rho\left(a_u \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} a_u^{-1}\right) \longrightarrow Id_V$$

Matrix $\Rightarrow$ char poly is continuous $\Rightarrow \rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$ has the same char poly as Id $\Rightarrow$ all eigenvalues are 1 $\Rightarrow$ unipotency.

Jacobson–Morozov lemma $\Rightarrow$ same for all nacpt. G. C' C 1

This finishes the proof of 1).

08.05.2018

Now we prove 2). We start with a technical Prop. from meas.-thy.

Prop: Assume G to be connected with no compact factors,

V to be a fin. dim. $\mathbb{R}$- or $\mathbb{C}$-vector space,

$\rho: G \longrightarrow GL(V)$ to be a cont. homomorphism

μ to be a $\rho(G)$-invariant vector space on $\mathbb{P}(V) = V \setminus 0 / v \sim \mathbb{K}v$

(note that ρ descends to an action on $\mathbb{P}(V)$, hence the above is valid)

such that $\mu(\mathbb{P}(V)) < \infty$.

Then μ is supported on $Fix(G) \overset{def}{=} \{[v] \in \mathbb{P}(V) \mid g[v] = [v] \quad \forall g \in G\}$.

PF OF PROP: Sts μ is supported on fixed points of

$$\{\rho(u) \mid u \in G,\ u \text{ unipotent}\},$$

just as in the pf. of 1). Recall that u unipt. $\Rightarrow \rho(u)$ is unipt.

$T = \rho(u) - Id$, $T^k \neq 0$ for some $k \in \mathbb{N}$, $T^{k+1} = 0$.

$\rho(u) T^k v = (Id + T) T^k v = T^k v \quad \Rightarrow [T^k v]$ is fixed by $\rho(u)$.

Moreover,

$$\rho(u^n)[v] = \rho(u)^n[v] = \left[\sum_{r=0}^{k}\binom{n}{r}T^r v\right] = \left[\binom{n}{k}^{-1}\sum_{r=0}^{k}\binom{n}{r}T^r v\right] \longrightarrow [T^k v]$$

(the last equality is projective space)

as $n \to \infty$.

PRT: μ-a.e. $[v] \in \mathbb{P}(V)$: $\exists (n_k)_{k\in\mathbb{N}}$ s.t. $\rho(u^{n_k})[v] \to [v]$.

$\Rightarrow$ μ-a.e. $[v] \in \mathbb{P}(V)$ is of the form $[T^k v]$, hence a fixed pt of $\rho(u)$.

Now we truly start proving 2).

Let W be a fixed subspace of $\rho(\Gamma)$, $d = \dim W$.

Consider the homomorphism $\hat{\rho}: G \to GL(\Lambda^d V)$ induced by ρ.

If W has basis $e_1, \dots, e_d$ then $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid \forall i_j \in \{1, \dots, d\}\}$ is a basis of $\Lambda^d W$ when subject to the relation $e_{i_1} \wedge \dots \wedge e_{i_j} \wedge e_{i_{j+1}} \wedge \dots \wedge e_{i_k} = -e_{i_1} \wedge \dots \wedge e_{i_{j+1}} \wedge e_{i_j} \wedge \dots e_{i_k}$

Since W is $\rho(\Gamma)$-invariant, $\Lambda^d W$ is a 1-dim invariant subspace of $\hat{\rho}(\Gamma)$.

$\to [\Lambda^d W] \in \mathbb{P}(\Lambda^d V)$ is a fixed pt of $\hat{\rho}(\Gamma) \curvearrowright \mathbb{P}(\Lambda^d V)$

We get a map $\overline{\rho}: G/\Gamma \to \mathbb{P}(\Lambda^d V)$

$$g\Gamma \longmapsto g[\Lambda^d W];$$

well-defdness follows from the above discussion.

We get a finite volume $\hat{\rho}(G)$-invariant measure $\overline{\nu} := \overline{\rho}_* \nu$.

By the Prop.: $\overline{\rho}(G/\Gamma) = \operatorname{supp}\overline{\nu} \subseteq \operatorname{Fix}(G)$

(first $=$: def. of $\overline{\nu}$; $\subseteq$: Prop.)

$\Rightarrow \overline{\rho}(e\Gamma) = [\Lambda^d W]$ fixed by $\forall g \in G$

$\Rightarrow$ W is fixed by $\forall g \in G$

~~Since the action is continuous, W is fixed by $\forall g \in G$.~~

For the remainder of lecture 8, suppose that G is connected with no cpt. factors.

Cor. $H<G$ connected closed subgp. s.t. $\gamma H \gamma^{-1} = H \ \forall \gamma \in \Gamma \Rightarrow H \lhd G$.

Pf: $\mathfrak{h}$: the Lie algebra of H (i.e. the tangent space at $e \in H$)

$\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$, the Lie algebra of G.

Γ normalises $H \Rightarrow \mathfrak{h}$ is invariant under $\mathrm{Ad}_G \Gamma = \{\text{aut. homom. } G \to GL(\mathfrak{g})\}$

Thm. 2) $\Rightarrow \mathfrak{h}$ is invariant under $\mathrm{Ad}_G G$

H closed, connected $\overset{\text{Lie theory}}{\rightsquigarrow} H \lhd G$. □

Cor. $C_G(\Gamma) \overset{\text{def}}{=} \{g \in G \mid g\gamma = \gamma g \ \forall \gamma \in \Gamma\} = Z(G) \overset{\text{def}}{=} \{g \in G \mid gh = hg \ \forall h \in G\}$.

Pf: G linear $\Rightarrow G \subseteq SL_\ell(\mathbb{R})$ for some $\ell \Rightarrow G \subseteq Mat_{\ell \times \ell}(\mathbb{R})$

Define ρ : $G \longrightarrow GL(V)$

$A \longmapsto gAg^{-1}$

If $c \in C_G(\Gamma) \Rightarrow \rho(\gamma)c = c \quad \forall \gamma \in \Gamma$

Thm. 1) $\Rightarrow \rho(g)c = c \quad \forall g \in G \Rightarrow c \in Z(G) \Rightarrow C_G(\Gamma) \subseteq Z(G)$.

The inclusion $C_G(\Gamma) \supseteq Z(G)$ always holds. □

Cor. $N \lhd \Gamma$ finite $\to N < Z(G)$. (Note that $N < Z(G) \Rightarrow N \lhd G$ always holds.)

Pf: $\Gamma \curvearrowright N$ by $\gamma \cdot n = \gamma n \gamma^{-1} \quad \forall \gamma \in \Gamma, n \in N$.

$\gamma \in \Gamma$ acts trivially on N iff $\gamma \in C_\Gamma(N)$

We get a map $\Gamma / C_\Gamma(N) \hookrightarrow \mathrm{Aut}(N)$.

By the finiteness assumption, $|\mathrm{Aut}(N)| \leq |N|! < \infty$

$\Rightarrow \Gamma / C_\Gamma(N)$ is finite $\Rightarrow |\Gamma : C_\Gamma(N)| < \infty$. Hence $C_\Gamma(N)$ is a lattice in Γ.

(Check all the properties as an EXERCISE.)

$N < C_G(C_\Gamma(N)) = Z(G)$ by the prev. Cor. □

Cor. $|N_G(\Gamma) : \Gamma| < \infty$.

Pf: Claim. $N_G(\Gamma)^\circ$ centralises Γ.

Γ discrete. Hence if $g\gamma g^{-1} \in \Gamma$ & g is close to e then $g\gamma g^{-1} = \gamma$.

(This could be made rigorous.) This proves the Claim.

$N_G(\Gamma)^\circ \subseteq C_G(\Gamma) = Z(G)$ is finite

$\Rightarrow N_G(\Gamma)^\circ$ is trivial. $\rightarrow N_G(\Gamma)$ is discrete.

$\Rightarrow N_G(\Gamma)$ has a strict fundamental domain in G

Set $F := \{f_i\}_{i\in I}$ as a set of coset representatives for $N_G(\Gamma)/\Gamma$.

Let $\mathcal{F}$ be a strict fund. domain for $N_G(\Gamma)/\Gamma$

$\Rightarrow \bigcup_{i\in I} f_i\mathcal{F}$ is a strict fund. domain for Γ.

$\Rightarrow F\cdot\mathcal{F}$ is of finite volume $\Rightarrow F$ is finite. $\Rightarrow |N_G(\Gamma):\Gamma| < \infty$ □

Zariski topology (intermezzo)

Def. $\mathbb{R}[x_{11},\dots,x_{\ell\ell}] \ni q$, $A \in Mat_{\ell\times\ell}(\mathbb{R})$, $q(A)$ is defined entry-wise, substituting $x_{ij} := a_{ij}$.

Def. $S \subseteq Mat_{\ell\times\ell}(\mathbb{R})$ is <u>Zariski-closed</u> if $\exists Q \subseteq \mathbb{R}[x_{11},\dots,x_{\ell\ell}]$ s.t.

$$S = \{A \in Mat_{\ell\times\ell}(\mathbb{R}) \mid q(A) = 0 \quad \forall q \in Q\}.$$

Ex. These form the closed sets of a topology on $Mat_{\ell\times\ell}(\mathbb{R})$.

Def. The <u>Zariski closure</u> of $X \subseteq Mat_{\ell\times\ell}(\mathbb{R})$ is its closure in the Zariski topology, denoted as $\overline{X}^Z$.

Ex. $SL_\ell(\mathbb{R})$ is Zariski-closed.

$H = \left\{ \begin{pmatrix} e^t & & & \\ & e^{-t} & & \\ & & 1 & t \\ & & 0 & 1 \end{pmatrix} \middle| \, t\in\mathbb{R} \right\}$ is not Zariski-closed, a polynomial does not recognize an exponential (non-precise)

$\overline{H}^Z = \left\{ \begin{pmatrix} 1/a & & & \\ & a & & \\ & & 1 & t \\ & & 0 & 1 \end{pmatrix} \middle| \, a,t\in\mathbb{R},\ a\neq 0 \right\}$ This is not the same as the closure in the usual topology.

Rmk. $\mathbb{R}[x_{11}, \dots, x_{ee}]$ is noetherian $\Rightarrow$ any Zariski-closed set can be described with finitely many polynomials

Thm. (Borel Density Theorem) Γ is Zariski-dense in G, i.e. $\overline{\Gamma}^Z = G$.

Pf: $\left(\overline{\Gamma}^Z\right)^0 < G$ Ex. check this. (Every operation is algebraic, polynomials are algebraic, taking the conn. component is continuous.)

$\left(\overline{\Gamma}^Z\right)^0$ is closed in G (Ex.)

Γ normalises $\left(\overline{\Gamma}^Z\right)^0 \Rightarrow G$ normalises $\left(\overline{\Gamma}^Z\right)^0 \Rightarrow \left(\overline{\Gamma}^Z\right)^0 = G$.

(G semis., Γ lattice)

□

Further properties of lattices (no proofs)

15.05.2018

Recall: For a set S, $S^{-1} := \{s^{-1} \mid s \in S\}$ where s^{-1} is just a formal symbol.

$F(S) = \{$all words on $S \amalg S^{-1}$ containing no substring of the form ss^{-1} or $s^{-1}s$, $s \in S\}$

Multiplication = concatenation followed by reduction of substrings ss^{-1}, $s^{-1}s$

$|S| = |S'| \Rightarrow F(S) \cong F(S')$, so we write $F_{|S|}$ for $F(S)$

Def. H group is finitely generated if $F_r \twoheadrightarrow H$ for some r.

H is finitely presented if in addition $\exists R \subset F_r$ s.t. $\ker(F_r \twoheadrightarrow H)$ is the smallest normal subgroup containing R.

Notation. H fin. pres. gp., S fin. set., $R \subset F(S)$ as above, i.e. $\widetilde{\langle\langle R \rangle\rangle} = \ker(F(S) \twoheadrightarrow G)$ (normally generated subgroup)

then we write $G = \langle S \mid R \rangle$.

Thm. (PO) Γ is finitely presented.

□

Def. A group H is torsion free if $\nexists n \in \mathbb{N}, h \in H : h^n = e$ for $h \neq e$.

$SL_2(\mathbb{Z})$ is not torsion free: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Thm. (Selberg's Lemma, PO) Γ has a torsion free finite index subgroup.

□

5. Arithmetic groups

We want to generalise $SL_n(\mathbb{Z}) < SL_n(\mathbb{R})$: given $G < SL_n(\mathbb{R})$ take $G \cap SL_n(\mathbb{Z})$.

Problem. $\exists B \in SL_n(\mathbb{Z})$ s.t. $BGB^{-1} \cap SL_n(\mathbb{Z}) = \{e\}$ Exc. 5.1.1.

Hence in order to get an interesting theory of groups we need to do something more: in particular, we need $\mathbb{Q}$ to get involved.

Prop. $W \subset \mathbb{R}^l$ subspace. Then TFAE:

1) $W \cap \mathbb{Z}^l$ is a cocompact lattice in W
2) W is spanned by $W \cap \mathbb{Z}^l$
3) $W \cap \mathbb{Q}^l$ is dense in W
4) W can be defined by linear equations over $\mathbb{Q}$.

Pf: 1) $\longrightarrow$ 2): $V = \mathrm{Span}_{\mathbb{R}}(W \cap \mathbb{Z}^l) < W \Rightarrow W/V = \mathbb{R}^d$ for some $d \in \mathbb{N}$

$W \cap \mathbb{Z}^l \subset V \Rightarrow$ we have a cont. surj. map $\underbrace{W/W\cap\mathbb{Z}^l}_{\text{cocpt.}} \longrightarrow W/V = \mathbb{R}^d$

$\Rightarrow d = 0$. by cocptness.

2) $\longrightarrow$ 1): $\{e_1, \dots, e_k\}$ basis of $\mathbb{R}^k$. $\exists T: \mathbb{R}^k \longrightarrow W$ lin. isom.

s.t. $\{T(e_j)\}_j \subseteq W \cap \mathbb{Z}^l \Rightarrow T(\mathbb{Z}^k) \subseteq W \cap \mathbb{Z}^l$

$\mathbb{R}^k/\mathbb{Z}^k$ is cpt., T is continuous $\Rightarrow W \cap \mathbb{Z}^l \subset W$ is a lattice

2) $\longrightarrow$ 3): $\mathbb{Q} \subset \mathbb{R}$ dense $\Rightarrow T(\mathbb{Q}^l)$ is dense

4) $\longrightarrow$ 2): W solution space to lin. eqs. over $\mathbb{Q}$

$\Rightarrow$ we can find a basis of vectors with $\mathbb{Q}$-coefficients

Multiply with denominators $\longrightarrow$ $\mathbb{Z}$-basis.

3) $\longrightarrow$ 4): $\mathbb{Q}^l \cap W \subset W$ dense

$W^\perp = \{x \in \mathbb{R}^l \mid wx = 0 \ \ \forall w \in W \cap \mathbb{Q}^l\}$, this is def'd by lin. equations /$\mathbb{Q}$

$\Rightarrow$ has a $\mathbb{Q}$-basis $\Rightarrow W = (W^\perp)^\perp$ is def'd by lin. equations /$\mathbb{Q}$.

Def. Let $H < SL(\ell, \mathbb{R})$ be a closed subgroup. We say that H is defined over $\mathbb{Q}$ if

$\exists\, Q \subseteq \mathbb{Q}[x_{11}, \dots, x_{\ell\ell}]$ s.t. • $\mathrm{Var}(Q) = \{ g \in SL(\ell, \mathbb{R}) \mid q(g) = 0 \ \ \forall q \in Q \} \leq SL(\ell, \mathbb{R})$ subgp.,

• $H^\circ = \mathrm{Var}(Q)^\circ_1$

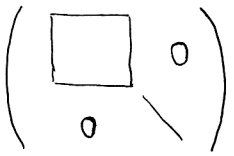
• H has only fin. many connected components.

In other words, H is commensurable to a group def'd by polynomials with $\mathbb{Q}$-coefficients.

Ex. 1) $SL(\ell, \mathbb{R})$, $Q = \emptyset$

2) $n < \ell$, $SL(n, \mathbb{R})$ in the top left corner of $SL(\ell, \mathbb{R})$,

$$Q = \{ x_{ij} - \delta_{ij} \mid \max(i, j) > n \}$$



3) $A \in SL(\ell, \mathbb{Q})$

$$SO_\ell(A, \mathbb{Q}) = \{ g \in SL(\ell, \mathbb{R}) \mid g A g^T = A \}$$

Rmk. By noetherianity, Q can always be chosen to be finite.

Prop. G is isogenous to a group defined over $\mathbb{Q}$.

Pf: Go over the list. □

Notation. $\mathcal{O} \subset \mathbb{R}$ ring, $1 \in \mathcal{O}$. Then define $G_{\mathcal{O}} := G \cap SL(\ell, \mathcal{O})$

Ex. $\varphi : SL(n, \mathbb{C}) \hookrightarrow SL(2n, \mathbb{R})$ embedding

Then $\varphi(SL(n, \mathbb{C}))_{\mathbb{Q}} = \varphi(SL(n, \mathbb{Q}[i]))$,

$$\varphi(SL(n, \mathbb{C}))_{\mathbb{Z}} = \varphi(SL(n, \mathbb{Z}[i]))$$

Prop. $H < SL(\ell, \mathbb{R})$ connected, almost Zariski closed subgp.

Then H is def'd over $\mathbb{Q}$ iff $H_{\mathbb{Q}} < H$ is dense.

Def. H is <u>almost Zariski closed</u> if

- H has finitely many connected components
- $\exists\, H_1 < SL(\ell, \mathbb{R})$ Zariski-closed s.t. $H^\circ = H_1^\circ$.

Thm. G def'd over $\mathbb{Q} \Rightarrow G_{\mathbb{Z}}$ lattice.

A lattice of the form $G_{\mathbb{Z}}$ is called arithmetic (not in the most general form yet).

For slightly more flexibility: we want the following

- G_1, G_2 $\mathbb{Q}$-groups, $\varphi: G_1 \longrightarrow G_2$ iso.
  If $\Gamma < G_1$ arithmetic then we want that $\varphi(\Gamma_1) < G_2$ is arithmetic.
- we want to ignore cpt factors
- Γ_1 comm. to Γ_2 and Γ_1 arithm. $\Rightarrow \Gamma_2$ arithm.

Def. $\Gamma < G$ <u>arithmetic</u> if

- $\exists\, G' < SL(\ell, \mathbb{R})$ closed conn. semisimple subgroup of $SL(n, \mathbb{R})$ for some n; G' def'd over $\mathbb{Q}$
- $\exists K < G$, $K' < G'$ compact normal subgps.
- $\exists \varphi: G^\circ/K \longrightarrow G'/K'$ iso s.t. $\varphi(\overline{\Gamma})$ is commensurable to $\overline{G'}_{\mathbb{Z}}$, where

$$\overline{\Gamma} = \frac{G^\circ \cap \Gamma}{\Gamma \cap K} \quad \text{and} \quad \overline{G}'_{\mathbb{Z}} = \frac{G'_{\mathbb{Z}}}{G'_{\mathbb{Z}} \cap K'}$$

($\overline{\cdot}$ does not mean Zariski closure)

Examples.

- $SL(\ell, \mathbb{Z})$
- $SO(A, \mathbb{R})_{\mathbb{Z}}$, $A \in SL(\ell, \mathbb{Q})$
- $\varphi: SL(n, \mathbb{C}) \xhookrightarrow{\sim} SL(2n, \mathbb{R})$.
  $\varphi\left(SL(n, \mathbb{Z}[i])\right) = \varphi\left(SL(n, \mathbb{C})\right)_{\mathbb{Z}}$
- $SL(2, \mathbb{Z}) < SL(2, \mathbb{R})$

  $N \in \mathbb{N}$: $\Gamma(N) = \left\{ g \in SL(2, \mathbb{Z}) \;\middle|\; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \begin{matrix} a \equiv d \equiv 1 \ (N) \\ b \equiv c \equiv 0 \ (N) \end{matrix} \right\} < SL(2, \mathbb{Z})$

  $\Gamma(N)$ is of finite index. To see this, consider the reduction $r_N: SL(2, \mathbb{Z}) \longrightarrow SL(2, \mathbb{Z}/N\mathbb{Z})$, and recognize that $\Gamma(N) = \ker(r_N)$.

Rmk.

- G has no cpt factors $\Rightarrow K$ is finite.
- Since there are only countably many finite subsets of $\mathbb{Q}[x_{11}, \ldots, x_{\ell\ell}]$, there are only countably many arithmetic lattices.

Two big theorems by Margulis

Thm. (Margulis)

If • G is not isogeneous to $SO(1,n)\times K$ or $SU(1,n)\times K$ for any cpt Lie group K;

• $\Gamma < G$ is an irreducible lattice,

then Γ is arithmetic.

Note: $SL(2,\mathbb{R})$ is isog. to $SO(1,2)$

$SL(2,\mathbb{C})$ is isog. to $SO(1,3)$ → the Margulis thm. does not apply

Recall: $Comm_G(\Gamma) = \{g\in G \mid \Gamma \cap g\Gamma g^{-1} < \Gamma \text{ is of finite index}\}$

Exc. If G is def'd over $\mathbb{Q}$ then $Comm_G(G_{\mathbb{Z}}) > G_{\mathbb{Q}}$ (EXERCISE 4.8.11)

So if Γ is arithmetic, G is connected and has no cpct factors then $Comm_G(\Gamma) < G$ is dense.

Thm. (Margulis, Commensurability criterion for arithmeticity)

$\Gamma < G$ is arithmetic iff $Comm_G(\Gamma) < G$ is dense.

Prop. (Godement compactness criterion)

G def'd over $\mathbb{Q}$. Then $G/G_{\mathbb{Z}}$ is cpt iff $G_{\mathbb{Z}}$ does not contain any nontrivial nilpotent elts.

Pf: $G/G_{\mathbb{Z}}$ cpt. ⇒ $G_{\mathbb{Z}}$ contains no nontrivial nilp. elts, this was already seen for lattices.

$G/G_{\mathbb{Z}}$ noncpt, WTS $\exists\, \gamma\in G_{\mathbb{Z}}$ nontrivial nilpotent

$G_{\mathbb{Z}}$ noncpt lattice ⇒ Id is an accumulation point of G_Γ.

⇒ $\exists g\in G,\ \gamma\in G_{\mathbb{Z}}$ s.t. ${}^g\gamma \approx Id$.

⇒ $char_\gamma(x) = char_{g\gamma}(x) \approx (x-1)^\ell$

⇒ $char_\gamma(x) = (x-1)^\ell$ ⇒ γ unipotent. □

Thm. G has no cpt factors, $\Gamma < G$ arithmetic.

Then G/Γ cpt. $\Leftrightarrow$ Γ has no nontriv unipotent elt.

Pf: Consequence of the previous Prop. (i.e. the GCC).

Prop If • $B: \mathbb{Q}^\ell \times \mathbb{Q}^\ell \longrightarrow \mathbb{Q}$ is a symmetric bilinear form,

• $B(x,x) \neq 0 \quad \forall x \in \mathbb{Q}^\ell \setminus 0$.

Then $SO(B)_{\mathbb{Z}} < SO(B)_{\mathbb{R}}$ is cocompact. where $SO(B) = \{g \in SL(\ell,\mathbb{R}) \mid B(gv,gw) = B(v,w) \ \forall v,w \in \mathbb{R}^\ell\}$*

Pf: We will use the Mahler compactness criterion.

Let $G := SO(B)_{\mathbb{R}}$, $\Gamma := G_{\mathbb{Z}}$, assume $B(\mathbb{Z}^\ell, \mathbb{Z}^\ell) \subseteq \mathbb{Z}$.

We can do this because $\widetilde{B} = \overline{\prod} \ d \cdot B$ is such a form

d denominator in the coeffs of B

and $SO(B) = SO(\widetilde{B})$.

* Technically B takes rational vectors as arguments, but we may $\otimes \mathbb{R}$.

1) Use MCC to show that $\pi(G)$ is precompact where

$\pi: SL(\ell,\mathbb{R}) \twoheadrightarrow SL(\ell,\mathbb{R})/SL(\ell,\mathbb{Z})$ is the projection map.

2) Show that $\pi(G)$ is closed, hence cpt.

3) Show that $G/\Gamma \longrightarrow SL(\ell,\mathbb{R})/SL(\ell,\mathbb{Z})$ is a homeo onto $\pi(G)$.

Since $\pi(G)$ is cpt, this finishes the proof.

1) $\exists g_n \in G, \ v_n \in \mathbb{Z}^\ell \setminus \{0\}$ s.t. $g_n v_n \longrightarrow 0$.

$B(\mathbb{Z}^\ell, \mathbb{Z}^\ell) \subseteq \mathbb{Z} \Rightarrow 1 \leq |B(v_n, v_n)| = |B(g_n v_n, g_n v_n)| \longrightarrow 0$ ↯

2) Suppose $g_n \gamma_n \longrightarrow h \in SL(\ell,\mathbb{R})$ for $g_n \in G$, $\gamma_n \in SL(\ell,\mathbb{Z})$.

WTS $h \in G \cdot SL(\ell,\mathbb{Z})$.

Let $\{e_1, \dots, e_\ell\}$ be the std. basis of $\mathbb{R}^\ell$.

$B(\gamma_n e_j, \gamma_n e_k) \in \mathbb{Z}$

$B(g_n \gamma_n e_j, g_n \gamma_n e_k) = B(\gamma_n e_j, \gamma_n e_k$

$\downarrow n \to \infty$

$B(he_j, he_k)$

$\exists n_0 \in \mathbb{N}$ s.t. $B(\gamma_n e_j, \gamma_n e_k) \overset{\text{Claim}}{=} B(h e_j, h e_k)$ $\forall n > n_0$ by discreteness

$(\gamma_n e_j)_j$ is a basis in $\mathbb{R}^\ell$ and $B((h\gamma_n^{-1})\gamma_n e_j, (h\gamma_n^{-1})\gamma_n e_k) = B(\gamma_n e_j, \gamma_n e_k)$,

implying the claimed equality above.

Thus $h\gamma_n^{-1} \in SO(B)$, as desired.

3) $\varphi: G/\Gamma \longrightarrow SL(\ell,\mathbb{R})/SL(\ell,\mathbb{Z})$ $\qquad$ G/Γ is a T2-space.

$g\Gamma \longmapsto g\,SL(\ell,\mathbb{Z})$

automatically bijective with compact image $\pi(G)$.

STS: $\varphi^{-1}: \pi(G) \longrightarrow G/\Gamma$ is continuous. (Point set topology fact:

if $f: X \to Y$ is bijective, continuous, X cpt, Y T2 $\Rightarrow$ f homeo.)

Suppose $\varphi(g_n\Gamma)$ converges. WTS $g_n\Gamma$ converges.

$\exists (\gamma_n)_n$ sequence in $SL(\ell,\mathbb{Z})$, $\exists h \in G$ s.t. $g_n\gamma_n \longrightarrow h$.

Use the proof of 2) (i.e. discreteness argument) $\Rightarrow$ $h \in G\gamma_n$ for $n \gg 1$.

$\Rightarrow$ $\gamma_n \in Gh = \underset{h\in G}{G}$. $\Rightarrow$ $\gamma_n \in \underbrace{G \cap SL(\ell,\mathbb{Z})}_{\Gamma}$ for $n \gg 1$

$\Rightarrow$ $g_n\Gamma \longrightarrow h\Gamma$, as desired. □

<u>Thm.</u> (Legendre) $a,b,c \in \mathbb{Z}\setminus\{0\}$ square-free, pwise rel. prime, do not all have the same sign. Then $\exists (x,y,z) \in \mathbb{Z}^3\setminus\{0\}$ s.t. $ax^2+by^2+cz^2=0$ 29.05.2018

iff $\exists u,v,w \in \mathbb{Z}$ s.t. $\quad -ab \equiv u^2 \pmod c$

$-bc \equiv v^2 \pmod a$

$-ac \equiv w^2 \pmod b$.

PO, not hard, see "A classical intro to modern number thy" by K. Ireland, M. Rosen, Ch. 17 § 3. □

<u>Ex.</u> $q_1(x,y,z) := x^2+y^2-z^2$ $\rightsquigarrow$ $q_1(1,0,1) = 0$, $\exists$ nontriv isotropic vector

$q_2(x,y,z) := 11x^2+7y^2-5z^2$ has no nontriv isotropic vectors:

$$-7\cdot 11 = -77 \equiv 3 \mod 5$$

Note that over $\mathbb{R}$ these two forms are congredient, so q_2 does have nontriv isotropic vectors, just not in $\mathbb{Z}^3$.

Fact. (to be proven later) $SO(B_1)_{\mathbb{Z}} < SO(B_1)_{\mathbb{R}} = SO(2,1)$ non-cocpt

$SO(B_2)_{\mathbb{Z}} < SO(B_2)_{\mathbb{R}} \cong SO(2,1)$ cocpt.

Basis-free definition of arithmetic groups

So far we have always implicitely chosen a basis.

Def. V a linear vector space

1) A $\mathbb{Q}$-subspace $V_{\mathbb{Q}}$ of V is called a **$\mathbb{Q}$-form of V** if the $\mathbb{R}$-lin. map

$V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow V$ is an isomorphism.

$v \otimes t \longmapsto tv$

2) A polynomial (see 5)) $f: V \longrightarrow \mathbb{R}$ is **defined over $\mathbb{Q}$** wrt. a $\mathbb{Q}$-form $V_{\mathbb{Q}}$ if $f(V_{\mathbb{Q}}) \leq \mathbb{Q}$.

3) A subgroup L of the additive group $V_{\mathbb{Q}}$ is called a **$\mathbb{Z}$-lattice** if
- L is finitely generated
- the $\mathbb{Q}$-linear map $L \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow V_{\mathbb{Q}}$ is an iso.

4) For a $\mathbb{Q}$-form $V_{\mathbb{Q}} \subset V$ let

$$\underline{\mathrm{End}(V)_{\mathbb{Q}}} := \{ A \in \mathrm{End}(V) \mid A(V_{\mathbb{Q}}) = V_{\mathbb{Q}} \}.$$

5) For a real vector space W a map $Q: W \longrightarrow \mathbb{R}$ is a **polynomial** if $\exists \varphi: \mathbb{R}^l \xrightarrow{\sim} W$ $\mathbb{R}$-linear isomorphism s.t. $Q \circ \varphi: \mathbb{R}^l \longrightarrow \mathbb{R}$ is a polynomial in the usual sense.

6) A subgroup $H < SL(V)$ is **defined over $\mathbb{Q}$** wrt a $\mathbb{Q}$-form $V_{\mathbb{Q}} \subset V$ if $\exists Q$ set of polynomials on $\mathrm{End}(V)$ s.t.
- $\forall q \in Q$ is defined over $\mathbb{Q}$ wrt. $V_{\mathbb{Q}}$
- $\mathrm{Var}(Q) = \{ g \in SL(V) \mid q(g) = 0 \ \forall q \in Q \}$ is a subgroup of $SL(V)$
- $\mathrm{Var}(Q)^\circ < H$ is of finite index.

Rmk. $V_{\mathbb{Q}} = \mathbb{Q}^n \leq \mathbb{R}^n$ yields our previous definition.

Lemma.[1] V real vector space with a $\mathbb{Q}$-form $V_{\mathbb{Q}}$.

Then $\exists \varphi: V \xrightarrow{\sim} \mathbb{R}^l$ $\mathbb{R}$-lin. iso s.t. $\varphi(V_{\mathbb{Q}}) = \mathbb{Q}^l$.

Furthermore if $L \subset V_{\mathbb{Q}}$ is any $\mathbb{Z}$-lattice then we may choose φ s.t. $\varphi(L) = \mathbb{Z}^l$.

2) A polynomial $f: \mathbb{R}^l \longrightarrow \mathbb{R}$ is defined over $\mathbb{Q}$ wrt. $\mathbb{Q}^l$ if the coefficients of f are in $\mathbb{Q}$.

Prop: $G < GL(V)$ defined over $\mathbb{Q}$ wrt. $V_{\mathbb{Q}}$. Then

1) For any $\mathbb{Z}$-lattice $\mathcal{L}$ in $V_{\mathbb{Q}}$:

$$G_{\mathcal{L}} = \{ g \in G \mid g\mathcal{L} = \mathcal{L} \}$$

is an arithmetic subgroup.

2) If $\mathcal{L}_1, \mathcal{L}_2$ are $\mathbb{Z}$-lattices then $G_{\mathcal{L}_1}$ and $G_{\mathcal{L}_2}$ are commensurable.

Lemma. 3) If $\mathcal{L}_1, \mathcal{L}_2 < V_{\mathbb{Q}}$ $\mathbb{Z}$-lattices then $\exists p \in \mathbb{Z} \setminus \{0\}$ s.t.

$$p\mathcal{L}_1 \leq \mathcal{L}_2, \quad p\mathcal{L}_2 \leq \mathcal{L}_1$$

PF OF LEMMA: linear algebra, EXC. □

PF OF PROP: use the Lemma. □

Restriction of scalars

Need some alg.n.t., PO.

Recall: algebraic number field, ring of integers $\mathcal{O}_F$

$\{\sigma_1, \dots, \sigma_r\}$ embeddings into $\mathbb{C}$

$$\Delta : \mathcal{O}_F \longrightarrow \mathbb{C}^r$$
$$x \longmapsto (\sigma_1(x), \dots, \sigma_r(x))$$

Exc. $SL(l, \mathbb{C}^r) \cong SL(l, \mathbb{C})^r$

$\Rightarrow SL(l, \mathcal{O}_F) \xrightarrow{\Delta} SL(l, \mathbb{C})^r$ monomorphism

Note that as $\mathcal{O} < \mathbb{C}$ is discrete, $\Delta(\Gamma)$ is discrete $\forall \Gamma < SL(l, \mathcal{O}_F)$

Goal: if $\Gamma = G_{\mathcal{O}_F}$ and G is defined over F then

$\Delta(\Gamma) < SL(l, \mathbb{C})^r$ is an arithmetic subgroup.

Ex. $\Gamma = SL(2, \mathbb{Z}[\sqrt{2}])$, $G = SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, $\sigma: \mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{C}$ the nontrivial embedding

$$\Delta : \Gamma \longrightarrow G$$
$$\gamma \longmapsto (\gamma, \sigma(\gamma))$$

Claim. $\Delta(\Gamma)$ is an irreducible arithmetic subgroup of G.

Pf: $F = \mathbb{Q}(\sqrt{2})$, $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$

$\{(1,1), (\sqrt{2}, -\sqrt{2})\}$ is a $\mathbb{Q}$-basis for $\Delta(F)$ and an $\mathbb{R}$-basis for $\mathbb{R}^2$

$\Rightarrow$ $\Delta(F)$ is a $\mathbb{Q}$-form for $\mathbb{R}^2$.

$\Rightarrow$ $\Delta(F^2) = \{(u, \sigma(u)) \in F^4 \mid u \in F^2\}$ is a $\mathbb{Q}$-form for $\mathbb{R}^4$

and $\Delta(\mathcal{O})$ is a $\mathbb{Z}$-lattice in $\Delta(F^2)$

Moreover G is defined over $\mathbb{Q}$ (Exc. 5.5.1, 5.5.2)

$\Rightarrow$ $G_{\Delta(\mathcal{O}^2)}$ is an arithmetic subgroup of G by the prev. Prop.

and $G_{\Delta(\mathcal{O}^2)} = \Delta(\Gamma)$.

Irreducibility: $\{e\} = \Delta(\Gamma) \cap \left(SL(2,\mathbb{R}) \times \{e\}\right)$ $\Rightarrow$ $\Delta(\Gamma)$ is irreducible. □

05.06.20

Addendum to this proof: the proof of discreteness is missing.

Discreteness follows from the fact that $N_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}} : \mathbb{Z}[\sqrt{2}] \longrightarrow \mathbb{Z}$.

Def. F an algebraic number field.

1) Two distinct embeddings $\sigma_1, \sigma_2 : F \hookrightarrow \mathbb{C}$ are called **equivalent** if $\sigma_1(x) = \overline{\sigma_2(x)}$ $\forall x \in F$.

2) A **place** of F is an equivalence class of embeddings $\sigma : F \hookrightarrow \mathbb{C}$.

- A place σ is called **real** if it contains a single embedding, i.e. $\sigma(F) \subseteq \mathbb{R}$.
- A place σ is called **complex** if it contains two embeddings, i.e. $\sigma(F) \not\subseteq \mathbb{R}$.

3) $S^\infty := \{\text{places of } F\}$

4) For $\sigma \in S^\infty$: $F_\sigma := \begin{cases} \mathbb{R} & \sigma \text{ real} \\ \mathbb{C} & \sigma \text{ complex} \end{cases}$

F_σ is called the **completion of F at σ**. (Note that $F \subset F_\sigma$ is dense.)

5) For $Q \subseteq F_\sigma[x_{11}, \dots, x_{ll}]$: $\mathrm{Var}_{F_\sigma}(Q) := \{g \in SL(l, F_\sigma) \mid q(g) = 0 \quad \forall q \in Q\}$

6) Suppose $G \subseteq SL(\ell, \mathbb{R})$ is defined over F, i.e. $\exists Q \subseteq F[x_{11}, \dots, x_{\ell\ell}]$ s.t. $G^\circ = \mathrm{Var}(Q)^\circ$.

Then $\forall \sigma \in S^\infty$ let $\underline{G^\sigma := \mathrm{Var}_{F_\sigma}(\sigma(Q))^\circ}$, called the <u>Galois conjugate of G at σ</u>.

(Note that G^σ is defined over $\sigma(F)$.)

<u>Prop.</u> If G is defined over an algebraic number field $F \subseteq \mathbb{R}$ and $\mathcal{O}_F$ is the ring of integers of F then $\exists \mathring{G}_{\mathcal{O}_F} < G_{\mathcal{O}_F}$ finite index subgroup such that $\mathring{G}_{\mathcal{O}_F}$ embeds as an arithmetic subgroup of $\prod_{\sigma \in S^\infty} G^\sigma$ via the natural embedding $\Delta : \gamma \mapsto (\sigma(\gamma))_{\sigma \in S^\infty}$.

Furthermore if G is simple then $\Delta(\mathring{G}_{\mathcal{O}_F})$ is irreducible.

We give a proof through examples.

<u>Ex.</u> 1) $G = SO(x^2 + y^2 - \sqrt{2}\, z^2, \mathbb{R}) \simeq SO(1,2)$

Then $G_{\mathbb{Z}[\sqrt{2}]}$ is a cocompact arithmetic subgroup of G.

Pf of Ex: $\sigma : \mathbb{Q}(\sqrt{2}) \hookrightarrow \mathbb{C}$ the nontrivial embedding

$\Gamma = G_{\mathbb{Z}[\sqrt{2}]}$

$K' := SO(x^2 + y^2 + \sqrt{2}\, z^2) = SO(3)$ compact.

So $\sigma(\Gamma) \subseteq K' \not\subseteq G$.

Recall that arithmetic groups are defined only up to quotients by compacts.

Define $\Delta : \Gamma \longrightarrow G \times K'$

$\gamma \longmapsto (\gamma, \sigma(\gamma))$

By the same argument as before: $\Delta(\Gamma) \subset G \times K'$ is an arithmetic subgroup.

K' cpt $\Rightarrow$ by modding out K' we obtain an arithm. subgroup $G_{\mathbb{Z}[\sqrt{2}]}$

It remains to show cocompactness.

We want to use Godement's compactness criterion, i.e. show that Γ has no nontrivial unipotent elements. Let $\gamma \in \Gamma \setminus \{e\}$.

$\Rightarrow \sigma(\gamma) \in K' \setminus \{e\}$ K' is compact $\Rightarrow \sigma(\gamma)$ is not unipotent

$\Rightarrow \exists \lambda \neq 1$ an eigenvalue of $\sigma(\gamma) \Rightarrow \gamma$ has eigenvalue $\sigma^{-1}(\lambda) \neq 1$

$\Rightarrow G_{\mathbb{Z}[\sqrt{2}]}$ is cpt by GCC. □

<u>Cor.</u> If G^σ is compact for some $\sigma \in S^\infty$ then $\Delta(G_{\mathcal{O}_F})$ is cocompact.

Ex. 2) $F = \mathbb{Q}(\sqrt[4]{2})$, $\mathcal{O}_F = \mathbb{Z}[\sqrt[4]{2}]$, $\Gamma = SL(2, \mathcal{O}_F)$, $G = SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \times SL(2,\mathbb{C})$

Claim: Γ embeds into G as an arithmetic subgroup.

Pf: Set $\alpha = \sqrt[4]{2}$. Its minimal polynomial is $x^4 - 2$ which has 4 distinct roots, determined by $\sigma_0(\alpha) = \alpha$, $\sigma_1(\alpha) = -\alpha$, $\sigma_2(\alpha) = i\alpha$, $\sigma_3(\alpha) = -i\alpha$

We get the embedding $\Delta : F \longrightarrow \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{C}$ restricting to $\Gamma \longrightarrow G$.

$$\gamma \longmapsto (\gamma, \sigma_1(\gamma), \sigma_2(\gamma))$$

$$N(x) = \prod \sigma_i(x) = \sigma_0(x)\,\sigma_1(x) \cdot |\sigma_2(x)|^2$$

Rmk. So far we did everything over $\mathbb{R}$, but they all work over $\mathbb{C}$ as well if the analogous conditions hold:

- $G \subseteq SL(\ell, \mathbb{C})$
- F is an alg. number field, $F \not\subseteq \mathbb{R}$
- G is defined over F, i.e. $\exists Q \subseteq F[x_{11}, \dots, x_{\ell\ell}]$ s.t. $G^\circ = \mathrm{Var}^\circ_{\mathbb{Q}}(G)$.

Ex. By doing the same as before we can show that $SO(n, \mathbb{Z}[i, \sqrt{2}])$ is an arithmetic lattice in $SO(n,\mathbb{C}) \times SO(n,\mathbb{C})$.

Prop. If $\Gamma = G_{\mathbb{Z}}$ is an irreducible lattice in G then there are

1) an alg number field F with completion F_∞
2) a connected simple subgroup $H < SL(\ell, F_\infty)$ for some ℓ such that H is defined over F.
3) an isogeny $\varphi : \prod_{\sigma \in S^\infty} H^\sigma \longrightarrow G$ s.t. $\varphi\left(\Delta\left(H_{\mathcal{O}_F}\right)\right)$ is commensurable to Γ.

This basically means that <u>all</u> irreducible arithmetic lattices come from a restriction of scalars.

Pf: To make life easier: $G \subseteq SL(n, \mathbb{C})$, $\Gamma = G_{\mathbb{Z}[i]}$, G defd over $\mathbb{Q}(i)$

Write $G = G_1 \times \dots \times G_r$, all G_i are simple, $H = G_1$.

If $r = 1$: set $F = \mathbb{Q}(i)$

$\Sigma := \mathrm{Gal}(\mathbb{C}/\mathbb{Q}(i))$. Since G is defined over $\mathbb{Q}(i)$, $\forall \sigma \in \Sigma$ fix G_i

σ permutes the factors G_i.

Claim: $\Sigma \curvearrowright \{G_i\}_{i=1}^r$ transitively.

Pf: $\lightning$ Suppose $\{G_{i_1}, \dots, G_{i_m}\}$ invariant under Σ. $i_1 < \dots < i_m$

$\Rightarrow A = G_{i_1} \times \dots \times G_{i_m}$ def'd over $\mathbb{Q}(i)$

$\Rightarrow A' = \prod_{j \notin \{i_1, \dots, i_m\}} G_j$ def'd over $\mathbb{Q}(i)$ (Exc. 5.5.4)

$\Rightarrow \Gamma = A_{\mathbb{Z}[i]} \times A'_{\mathbb{Z}[i]}$ where $A_{\mathbb{Z}[i]} \subset A$, $A'_{\mathbb{Z}[i]} \subset A'$ arithmetic

$\Rightarrow \Gamma$ is not irreducible. $\lightning$ ©

Set $\Sigma_1 = \{\sigma \in \Sigma \mid \sigma(H) = H\}$. Since the above action is transitive, $[\Sigma : \Sigma_1] = r$.

$\Rightarrow F := \{z \in \mathbb{C} \mid \sigma(z) = z \ \forall \sigma \in \Sigma_1\}$ is a degree r field extension of $\mathbb{Q}(i)$

and H is defined over F as $\sigma(H) = H \ \forall \sigma \in \Sigma_1$.

Choose coset representatives $\sigma_1, \dots, \sigma_r$ of $\Sigma_1 < \Sigma$.

Then $\sigma_1|_F, \dots, \sigma_r|_F$ are the r places of F. After renumbering: $\sigma_j(H) = G_j$

$\Rightarrow \prod_{\sigma \in S^\infty} H^\sigma = H^{\sigma_1|_F} \times \dots \times H^{\sigma_r|_F} = G_1 \times \dots \times G_r$

For $h \in H_F$ let $\Delta^1(h) := \prod_{j=1}^r \sigma_j(h)$.

$\sigma(\Delta^1(h)) = \Delta^1(h) \quad \forall \sigma \in \Sigma \quad \rightarrow \quad \Delta^1(h) \in G_{\mathbb{Q}(i)}$

In fact, $\Delta^1(H) = G_{\mathbb{Q}(i)}$. This follows from the transitivity of the action.

$\Delta(H_{O_F})$ is commensurable to $\Gamma = G_{\mathbb{Z}[i]}$. □

Cor.

Arithmetic subgroups of $SL(2,\mathbb{R})$ (Ch. 6.)

Def. $SO(A,F) := \{ g \in SL(\ell, F) \mid g^T A g = A \}$ for $A \in Mat_{\ell\times\ell}(F)$ symmetric invertible

Def. $SO(B,F) := \{ g \in SL(\ell, F) \mid B(gv, gw) = B(v,w) \ \forall v,w \in F^\ell \}$

where B is a non-degenerate bilinear form $(\forall v \in F^\ell \setminus \{0\} \ \exists w \in F^\ell : B(v,w) \neq 0)$

Examples. 1) $SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ non-cocpt arith subgroup ($\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is unipotent)

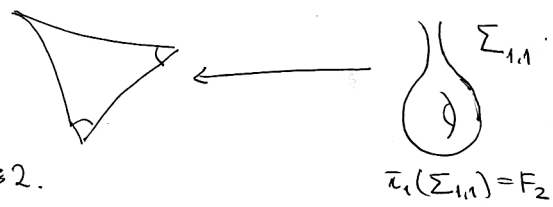
Rmk. We will prove teadt up to conjugation and commensurability this is the only non-cocpt arith subgroup of $SL(2,\mathbb{R})$.

However, there are a lot of interesting finite index subgroups,

e.g. $\exists \Gamma' < SL(2,\mathbb{R})$ fin. index s.t. $\Gamma \simeq F_2$

$F_r < F_2$ is of fin index $\forall r \geq 2$

$\Rightarrow F_r$ is a fin index subgp in $SL(2,\mathbb{R})$ $\forall r \geq 2$.



Every graph has a free fundamental group because by contracting the spanning tree we get a htp equivalence to a wedge of circles.

We can construct all fin. gen. non-ab free gp this way.

2) $a,b \in \mathbb{Z}_{>0}$, $G := SO(ax^2 + by^2 - z^2, \mathbb{R}) \simeq SO(2,1)$ isogenous to

Recall: if $(0,0,0)$ is the only integral solution to $ax^2 + by^2 = z^2$, then $G_{\mathbb{Z}}$ is a cocpt arith subgp of G

3) Restriction of scalars

- $F \neq \mathbb{Q}$ totally real number field, e.g. $\mathbb{Q}[\sqrt{2}]$
- $a,b \in F^+$ s.t. $\sigma(a) < 0$, $\sigma(b) < 0$ $\forall \sigma \in S^\infty \setminus \{id\}$
- $\mathcal{O}_F$ ring of integers
- $G := SO(ax^2 + by^2 - z^2, \mathbb{R}) \simeq SO(2,1)$

$\Rightarrow G_{\mathcal{O}_F}$ is a cocpt arith subgp of G

4) $h := diag(\sqrt{a}, \sqrt{b}, 1) \Rightarrow h^{-1} G h = SO(2,1)$

Prop. Up to commensurability and conjugation 2) and 3) are the only (opt) nbhds of $SL(2,\mathbb{R})$.

Pf: Step 1: we know that there are

- an alg number field $F \subset \mathbb{R}$ with ring of integers $\mathcal{O}_F$
- $H \subseteq SL(2,\mathbb{R})$ simple, connected, defined over F
- an isogeny $\varphi: H \longrightarrow SL(2,\mathbb{R})$ s.t. $\varphi(H_{\mathcal{O}_F})$ is commensurable to our arithmetic lattice Γ.

Goal: $H \simeq SO(B,F)$ where B is a bilinear form of the form 2) or 3).

We need some Lie theory (see App. A.6)

$\mathfrak{h}$:= Lie algebra of $H = T_e H$ (e.g. the Lie algebra of $SL(2,\mathbb{R})$ is $\mathfrak{sl}(2,\mathbb{R}) = \{X \in Mat_{2\times 2}(\mathbb{R}) \mid tr(X)=0\}$)

= Lie algebra of $SO(2,1)$

$\simeq \mathbb{R}^3$

$H \curvearrowright \mathfrak{h}$ by the adjoint representation:

$Ad(g) := \mathfrak{h} \longrightarrow \mathfrak{h}$ defined by $Ad(g)(v) := (d\psi_g)(v)$

where $\psi_g : H \longrightarrow H$ is given by $\psi_g(h) = ghg^{-1}$.

Computation shows that $Ad(g)(X) = gXg^{-1} \quad \forall g \in H \quad \forall X \in \mathfrak{h}$

The Lie algebra $\mathfrak{h}$ comes with a Lie bracket

$[-,-]: \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathfrak{h}$ defined by $[X,Y] = XY - YX \quad \forall X,Y \in \mathfrak{h}$

This gives us a representation $ad: \mathfrak{h} \longrightarrow End\,\mathfrak{h}$ given by $ad(X)(Y) = [X,Y]$.

We obtain a bilinear form called the Killing form:

$$\kappa: \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{R}$$
$$(X,Y) \longmapsto tr\left(ad(X) \cdot ad(Y)\right)$$

Fact: $Ad(g)$ preserves κ $\forall g \in H$ and Ad is an isogeny.

Step 2: NTS κ has the correct form.

Write $\kappa(x,x) = ax_1^2 + bx_2^2 + cx_3^2$. Need: $a,b,\bar{c} \in F^+$

Know: $SO(\kappa,\ell)$ is isog. to $SO(2,1)$ $\rightarrow$ get: $a,b,-c \in F^+$.

Divide by c $\rightarrow$ $\kappa(x,x) = ax_1^2 + bx_2^2 - x_3^2$

Step 3. $\forall \sigma \neq id : \sigma(a)\,\sigma(b) < 0$

Know: $\Delta(H_{\mathbb{Q}})$ irred lattice in $\prod_{\sigma \in S^\infty} H^\sigma$ and proj. in H^{id} lattice

$\Rightarrow$ H^σ cpt $\forall \sigma \neq id$ $\rightarrow$ $H^\sigma \simeq SO(2,1)$ $\forall \sigma \neq id$

$\Rightarrow$ $\sigma(a), \sigma(b), \sigma(-1)$ have the same sign $\Rightarrow$ $\sigma(a), \sigma(b) < 0$. □

Prop: $SL(2,\mathbb{Z})$ is the only non-cpt arith subgp of $SL(2,\mathbb{R})$ up to conjugation and commensurability.

Pf: Work with $SO(2,1)$. Goal: prove that $SO(2,1)_{\mathbb{Z}}$ is the "only" non-cpt arith subgp of $SO(2,1)$.

Know: $\exists G < SL(\ell, \mathbb{R})$ simple for some ℓ s.t. G is defined over some number field F and $\exists \varphi : G \longrightarrow SO(2,1)$ isogeny s.t. $\varphi(G_{\mathcal{O}_F})$ is comm to Γ.

Pf of prev. Prop $\Rightarrow$ $G = SO(B,\mathbb{R})$ where $B : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ bilinear form of signature $(2,1)$.

NTS: G is defined over $\mathbb{Q}$.

Recall from the original def of arith gps: $\exists K'$ cpt s.t.

$G' = G \times K' < SL(\ell, \mathbb{R})$, G' defined over $\mathbb{Q}$, $\Gamma K' = G'_{\mathbb{Z}} K'$

N = Zariski closure of unipotent elts in $G'_{\mathbb{Z}}$

K' cpt $\Rightarrow$ $N \subseteq G$

N is normalised by $G'_{\mathbb{Z}} \xrightarrow[\text{density}]{\text{Borel}} N \lhd G \xrightarrow{G \text{ simple}} N = G$.

G is the Zariski closure of a group defined over $\mathbb{Z}$ $\Rightarrow$ G is def. /$\mathbb{Q}$. □

Step 2: Have: $G = SO(B,\mathbb{R})$. Want: $B(x,x) = x_1^2 + x_2^2 - x_3^2$

Γ not cocpt $\Rightarrow$ B is isotropic over F $\Rightarrow$ $\exists u \in F^3 \setminus \{0\}$: $B(u,u) = 0$.

Choose $v \in F^3 \setminus \{0\}$ s.t. $B(u,v) \neq 0$

By adding a multiple of u wma $B(v,v) = 0$.

Choose $w \in F^3 \setminus \{0\}$ s.t. $B(w,u) = B(w,v) = 0$.

Multiplying B with scalars preserves everything $\Rightarrow$ wma $B(w,w) = 2B(v,v) = 1$.

$\rightarrow$ B has the desired form wrt. the basis $\{w, u+v, u-v\}$. □